Homotopy of unitaries in simple C^*-algebras with tracial rank one

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Abstract

Let $\epsilon > 0$ be a positive number. Is there a number $\delta > 0$ satisfying the following? Given any pair of unitaries u and v in a unital simple C^*-algebra A with $[v] = 0$ in $K_1(A)$ for which

$$\|uv - vu\| < \delta,$$

there is a continuous path of unitaries $\{v(t) : t \in [0,1]\} \subset A$ such that

$$v(0) = v, \;\; v(1) = 1 \;\text{ and }\; \|uv(t) - v(t)u\| < \epsilon \;\text{ for all }\; t \in [0,1].$$

An answer is given to this question when A is assumed to be a unital simple C^*-algebra with tracial rank no more than one. Let C be a unital separable amenable simple C^*-algebra with tracial rank no more than one which also satisfies the UCT. Suppose that $\phi : C \to A$ is a unital monomorphism and suppose that $v \in A$ is a unitary with $[v] = 0$ in $K_1(A)$ such that v almost commutes with ϕ. It is shown that there is a continuous path of unitaries $\{v(t) : t \in [0,1]\}$ in A with $v(0) = v$ and $v(1) = 1$ such that the entire path $v(t)$ almost commutes with ϕ, provided that an induced Bott map vanishes. Other versions of the so-called Basic Homotopy Lemma are also presented.

1 Introduction

Fix a positive number $\epsilon > 0$. Can one find a positive number δ such that, for any pair of unitary matrices u and v with $\|uv - vu\| < \delta$, there exists a continuous path of unitary matrices $\{v(t) : t \in [0,1]\}$ for which $v(0) = v$, $v(1) = 1$ and $\|uv(t) - v(t)u\| < \epsilon$ for all $t \in [0,1]$? The answer is negative in general. A Bott element associated with the pair of unitary matrices may appear. The hidden topological obstruction can be detected in a limit process. This was first found by Dan Voiculescu ([43]). On the other hand, it has been proved that there is such a path of unitary matrices if an additional condition, $\text{bott}_1(u, v) = 0$, is provided (see, for example, [2] and also 3.12 of [29]).

It was recognized by Bratteli, Elliott, Evans and A. Kishimoto ([2]) that the presence of such continuous path of unitaries in general simple C^*-algebras played an important role in the study of classification of simple C^*-algebras and perhaps plays important roles in some other areas. They proved what they called the Basic Homotopy Lemma: For any $\epsilon > 0$, there exists $\delta > 0$ satisfying the following: For any pair of unitaries u and v in A with $sp(u)$ δ-dense in $\mathbb{T}$ and $[v] = 0$ in $K_1(A)$ for which

$$\|uv - vu\| < \delta \;\text{ and }\; \text{bott}_1(u, v) = 0,$$

there exists a continuous path of unitaries $\{v(t) : t \in [0,1]\} \subset A$ such that

$$v(0) = v, \;\; v(1) = 1_A \;\text{ and }\; \|v(t)u - uv(t)\| < \epsilon$$

for all $t \in [0,1]$, where A is a unital purely infinite simple C^*-algebra or a unital simple C^*-algebra with real rank zero and stable rank one. Define $\phi : C(\mathbb{T}) \to A$ by $\phi(f) = f(u)$ for all

$f \in C(\mathbb{T})$. Instead of considering a pair of unitaries, one may consider a unital homomorphism from $C(\mathbb{T})$ into A and a unitary $v \in A$ for which v almost commutes with ϕ.

In the study of asymptotic unitary equivalence of homomorphisms from an AH-algebra to a unital simple C^*-algebra, as well as the study of homotopy theory in simple C^*-algebras, one considers the following problem: Suppose that X is a compact metric space and ϕ is a unital homomorphism from $C(X)$ into a unital simple C^*-algebra A. Suppose that there is a unitary $u \in A$ with $[u] = 0$ in $K_1(A)$ and u almost commutes with ϕ. When can one find a continuous path of unitaries $\{u(t) : t \in [0,1]\} \subset A$ with $u(0) = u$ and $u(1) = 1$ such that $u(t)$ almost commutes with ϕ for all $t \in [0,1]$?

Let C be a unital AH-algebra and let A be a unital simple C^*-algebra. Suppose that $\phi, \psi : C \to A$ are two unital monomorphisms. Let us consider the question when ϕ and ψ are asymptotically unitarily equivalent, i.e., when there is a continuous path of unitaries $\{w(t) : t \in [0,\infty)\} \subset A$ such that

$$\lim_{t\to\infty} w(t)^*\phi(c)w(t) = \psi(c) \text{ for all } c \in C.$$

When A is assumed to have tracial rank zero, it was proved in [31] that they are asymptotically unitarily equivalent if and only if $[\phi] = [\psi]$ in $KK(C,A)$, $\tau \circ \phi = \tau \circ \psi$ for all tracial states τ of A and a rotation map associated with ϕ and ψ is zero. Apart from some direct applications, this result plays crucial roles in the study of the problem to embed crossed products into unital simple AF-algebras and in the classification of simple amenable C^*-algebras which do not have the finite tracial rank (see [44], [32] and [33]). One of the key machinery in the study of the above mentioned asymptotic unitary equivalence is the so-called Basic Homotopy Lemma concerning a unital monomorphism ϕ and a unitary u which almost commutes with ϕ.

In this paper, we study the case that A is no longer assumed to have real rank zero, or tracial rank zero. The result of W. Winter in [44] provides the possible classification of simple finite C^*-algebras far beyond the cases of finite tracial rank. However, it requires to understand much more about asymptotic unitary equivalence in those unital separable simple C^*-algebras which have been classified. An immediate problem is to give a classification of monomorphisms (up to asymptotic unitary equivalence) from a unital separable simple AH-algebra into a unital separable simple C^*-algebra with tracial rank one. For that goal, it is paramount to study the Basic Homotopy Lemmas in a simple separable C^*-algebras with tracial rank one. This is the main purpose of this paper.

A number of problems occur when one replaces C^*-algebras of tracial rank zero by those of tracial rank one. First, one has to deal with contractive completely positive linear maps from $C(X)$ into a unital C^*-algebra C with the form $C([0,1], M_n)$ which are *not* homomorphisms but almost multiplicative. Such problem is already difficult when $C = M_n$ but it has been proved that these above mentioned maps are close to homomorphisms if the associated K-theoretical data of these maps are consistent with those of homomorphisms. It is problematic when one tries to replace M_n by $C([0,1], M_n)$. In addition to the usual K-theory and trace information, one also has to handle the maps from $U(C)/CU(C)$ to $U(A)/CU(A)$, where $CU(C)$ and $CU(A)$ are the closure of the subgroups of $U(C)$ and $U(A)$ generated by commutators, respectively. Other problems occur because of lack of projections in C^*-algebras which are not of real rank zero.

The main theorem is stated as follows: Let C be a unital separable simple amenable C^*-algebra with tracial rank one which satisfies the Universal Coefficient Theorem. For any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset C$, there exists $\delta > 0$, a finite subset $\mathcal{G} \subset C$ and a finite subset $\mathcal{P} \subset \underline{K}(C)$ satisfying the following:

Suppose that A is a unital simple C^*-algebra with tracial rank no more than one, suppose

that $\phi : C \to A$ is a unital homomorphism and $u \in U(A)$ such that

$$\|[\phi(c),\, u]\| < \delta \ \text{ for all } \ c \in \mathcal{G} \ \text{ and } \ \mathrm{Bott}(\phi, u)|_{\mathcal{P}} = 0. \tag{e 1.1}$$

Then there exists a continuous path of unitaries $\{u(t) : t \in [0,1]\} \subset A$ such that

$$u(0) = u, \ \ u(1) = 1 \ \text{ and } \ \|[\phi(c),\, u(t)]\| < \epsilon \ \text{ for all } \ c \in \mathcal{F} \tag{e 1.2}$$

and for all $t \in [0,1]$.

We also give the following Basic Homotopy Lemma in simple C^*-algebra with tracial rank one (see 6.3 below) :

Let $\epsilon > 0$ and let $\Delta : (0,1) \to (0,1)$ be a nondecreasing map. We show that there exists $\delta > 0$ and $\eta > 0$ (which does not depend on Δ) satisfying the following:

Given any pair of unitaries u and v in a unital simple C^*-algebra A with tracial rank no more than one such that $[v] = 0$ in $K_1(A)$,

$$\|[u,\, v]\| < \delta, \ \ \mathrm{bott}_1(u, v) = 0 \ \text{ and } \ \mu_{\tau\circ\imath}(I_a) \geq \Delta(a)$$

for all open arcs I_a with length $a \geq \eta$, there exists a continuous path of unitaries $\{v(t) : t \in [0,1]\} \subset A$ such that

$$v(0) = v, \ \ v(1) = 1 \ \text{ and } \ \|[u,\, v(t)]\| < \epsilon \ \text{ for all } \ t \in [0,1],$$

where $\imath : C(\mathbb{T}) \to A$ is the homomorphism defined by $\imath(f) = f(u)$ for all $f \in C(\mathbb{T})$ and $\mu_{\tau\circ\imath}$ is the Borel probability measure induced by the state $\tau \circ \imath$. It should be noted that, unlike the case that A has real rank zero, the length of $\{v(t)\}$ can not be controlled. In fact, it could be as long as one wishes.

In a subsequent paper, we use the main homotopy result (Theorem 8.4) of this paper and the results in [34] to establish a K-theoretical necessary and sufficient condition for homomorphisms from unital simple AH-algebras into a unital separable simple C^*-algebra with tracial rank no more than one to be asymptotically unitarily equivalent which, in turn, combining with a result of W. Winter, provides a classification theorem for a class of unital separable simple amenable C^*-algebras which properly contains all unital separable simple amenable C^*-algebras with tracial rank no more than one which satisfy the UCT as well as some projectionless C^*-algebras such as the Jiang-Su algebra.

2 Preliminaries and notation

2.1. Let A be a unital C^*-algebra. Denote by $\mathrm{T}(A)$ the tracial state space of A and denote by $\mathrm{Aff}(\mathrm{T}(A))$ the set of affine continuous functions on $\mathrm{T}(A)$.

Let $C = C(X)$ for some compact metric space X and let $L : C \to A$ be a unital positive linear map. Denote by $\mu_{\tau\circ L}$ the Borel probability measure induced by the state $\tau \circ L$, where $\tau \in \mathrm{T}(A)$.

2.2. Let a and b be two elements in a C^*-algebra A and let $\epsilon > 0$ be a positive number. We write $a \approx_\epsilon b$ if $\|a - b\| < \epsilon$. Let $L_1, L_2 : A \to C$ be two maps from A to another C^*-algebra C and let $\mathcal{F} \subset A$ be a subset. We write

$$L_1 \approx_\epsilon L_2 \ \text{ on } \ \mathcal{F},$$

if $L_1(a) \approx_\epsilon L_2(a)$ for all $a \in \mathcal{F}$.

Suppose that $B \subset A$. We write $a \in_\epsilon B$ if there is an element $b \in B$ such that $\|a - b\| < \epsilon$.

Let $\mathcal{G} \subset A$ be a subset. We say L is ϵ-$\mathcal{G}$-multiplicative if, for any $a, b \in \mathcal{G}$,

$$L(ab) \approx_\epsilon L(a)L(b)$$

for all a, $b \in \mathcal{G}$.

2.3. Let A be a unital C^*-algebra. Denote by $U(A)$ the unitary group of A. Denote by $U_0(A)$ the normal subgroup of $U(A)$ consisting of those unitaries in the path connected component of $U(A)$ containing the identity. Let $u \in U_0(A)$. Define

$$\mathrm{cel}_A(u) = \inf\{\mathrm{length}(\{u(t)\}) : u(t) \in C([0,1], U_0(A)), u(0) = u \text{ and } u(1) = 1_A\}.$$

We use $\mathrm{cel}(u)$ if the C^*-algebra A is not in question.

2.4. Denote by $CU(A)$ the *closure* of the subgroup generated by the commutators of $U(A)$. For $u \in U(A)$, we will use $\bar{u}$ for the image of u in $U(A)/CU(A)$. If $\bar{u}, \bar{v} \in U(A)/CU(A)$, define

$$\mathrm{dist}(\bar{u}, \bar{v}) = \inf\{\|x - y\| : x, y \in U(A) \text{ such that } \bar{x} = \bar{u}, \bar{y} = \bar{v}\}.$$

If $u, v \in U(A)$, then

$$\mathrm{dist}(\bar{u}, \bar{v}) = \inf\{\|uv^* - x\| : x \in CU(A)\}.$$

2.5. Let A and B be two unital C^*-algebras and let $\phi : A \to B$ be a unital homomorphism. It is easy to check that ϕ maps $CU(A)$ to $CU(B)$. Denote by $\phi^\ddagger$ the homomorphism from $U(A)/CU(A)$ into $U(B)/CU(B)$ induced by ϕ. We also use $\phi^\ddagger$ for the homomorphism from $U(M_k(A))/CU(M_k(A))$ into $U(M_k(B))/CU(M_k(B))$ $(k = 1, 2, ...)$.

2.6. Let A and C be two unital C^*-algebras and let $F \subset U(C)$ be a subgroup of $U(C)$. Suppose that $L : F \to U(A)$ is a homomorphism for which $L(F \cap CU(C)) \subset CU(A)$. We will use $L^\ddagger : F/CU(C) \to U(A)/CU(A)$ for the induced map.

2.7. Let A and B be in 2.6, let $1 > \epsilon > 0$ and let $\mathcal{G} \subset A$ be a subset. Suppose that L is ϵ-$\mathcal{G}$-multiplicative unital completely positive linear map. Suppose that u, $u^* \in \mathcal{G}$. Define $\langle L\rangle(u) = L(u)L(u^*u)^{-1/2}$.

Definition 2.8. Let A and B be two unital C^*-algebras. Let $h : A \to B$ be a homomorphism and let $v \in U(B)$ such that

$$h(g)v = vh(g) \text{ for all } g \in A.$$

Thus we obtain a homomorphism $\bar{h} : A \otimes C(S^1) \to B$ by $\bar{h}(f \otimes g) = h(f)g(v)$ for $f \in A$ and $g \in C(S^1)$. From the following splitting exact sequence:

$$0 \to SA \to A \otimes C(S^1) \leftrightarrows A \to 0 \tag{e 2.3}$$

and the isomorphisms $K_i(A) \to K_{1-i}(SA)$ $(i = 0, 1)$ given by the Bott periodicity, one obtains two injective homomorphisms:

$$\beta^{(0)} : K_0(A) \to K_1(A \otimes C(S^1)) \tag{e 2.4}$$
$$\beta^{(1)} : K_1(A) \to K_0(A \otimes C(S^1)). \tag{e 2.5}$$

Note, in this way, one can write $K_i(A \otimes C(S^1)) = K_i(A) \bigoplus \beta^{(1-i)}(K_{1-i}(A))$. We use $\widehat{\beta^{(i)}}$: $K_i(A \otimes C(S^1)) \to \beta^{(1-i)}(K_{1-i}(A))$ for the the projection to the summand $\beta^{(1-i)}(K_{1-i}(A))$.

For each integer $k \geq 2$, one also obtains the following injective homomorphisms:

$$\beta_k^{(i)} : K_i(A, \mathbb{Z}/k\mathbb{Z}) \to K_{1-i}(A \otimes C(S^1), \mathbb{Z}/k\mathbb{Z}), i = 0, 1. \tag{e 2.6}$$

Thus we write

$$K_{1-i}(A\otimes C(S^1),\mathbb{Z}/k\mathbb{Z})=K_{1-i}(A,\mathbb{Z}/k\mathbb{Z})\bigoplus\beta_k^{(i)}(K_i(A,\mathbb{Z}/k\mathbb{Z})),\ \ i=0,1. \tag{e 2.7}$$

Denote by $\widehat{\beta_k^{(i)}}:K_i(A\otimes C(S^1),\mathbb{Z}/k\mathbb{Z})\to\beta_k^{(1-i)}(K_{1-i}(A,\mathbb{Z}/k\mathbb{Z}))$ similarly to that of $\widehat{\beta^{(i)}}$., $i=1,2$. If $x\in\underline{K}(A)$, we use $\boldsymbol{\beta}(x)$ for $\beta^{(i)}(x)$ if $x\in K_i(A)$ and for $\beta_k^{(i)}(x)$ if $x\in K_i(A,\mathbb{Z}/k\mathbb{Z})$. Thus we have a map $\boldsymbol{\beta}:\underline{K}(A)\to\underline{K}(A\otimes C(S^1))$ as well as $\widehat{\boldsymbol{\beta}}:\underline{K}(A\otimes C(S^1))\to\boldsymbol{\beta}(\underline{K}(A))$. Thus one may write $\underline{K}(A\otimes C(S^1))=\underline{K}(A)\bigoplus\boldsymbol{\beta}(\underline{K}(A))$.

On the other hand $\bar{h}$ induces homomorphisms $\bar{h}_{*i,k}:K_i(A\otimes C(S^1)),\mathbb{Z}/k\mathbb{Z})\to K_i(B,\mathbb{Z}/k\mathbb{Z})$, $k=0,2,...$, and $i=0,1$. We use $\mathrm{Bott}(h,v)$ for all homomorphisms $\bar{h}_{*i,k}\circ\beta_k^{(i)}$. We write

$$\mathrm{Bott}(h,v)=0,$$

if $\bar{h}_{*i,k}\circ\beta_k^{(i)}=0$ for all $k\geq 1$ and $i=0,1$.

We will use $\mathrm{bott}_1(h,v)$ for the homomorphism $\bar{h}_{1,0}\circ\beta^{(1)}:K_1(A)\to K_0(B)$, and $\mathrm{bott}_0(h,u)$ for the homomorphism $\bar{h}_{0,0}\circ\beta^{(0)}:K_0(A)\to K_1(B)$.

Since A is unital, if $\mathrm{bott}_0(h,v)=0$, then $[v]=0$ in $K_1(B)$.

For a fixed finite subset $\mathcal{P}\subset\underline{K}(A)$, there exists $\delta>0$ and a finite subset $\mathcal{G}\subset A$ such that, if $v\in B$ is a unitary for which

$$\|h(a)v-vh(a)\|<\delta\ \text{ for all }\ a\in\mathcal{G},$$

then $\mathrm{Bott}(h,v)|_{\mathcal{P}}$ is well defined. In what follows, whenever we write $\mathrm{Bott}(h,v)|_{\mathcal{P}}$, we mean that δ is sufficiently small and $\mathcal{G}$ is sufficiently large so it is well defined.

Now suppose that A is also amenable and $K_i(A)$ is finitely generated ($i=0,1$). For example, $A=C(X)$, where X is a finite CW complex. When $K_i(A)$ is finitely generated, $\mathrm{Bott}(h,v)|_{\mathcal{P}_0}$ defines $\mathrm{Bott}(h,v)$ for some sufficiently large finite subset $\mathcal{P}_0$. In what follows such $\mathcal{P}_0$ may be denoted by $\mathcal{P}_A$. Suppose that $\mathcal{P}\subset\underline{K}(A)$ is a larger finite subset, and $\mathcal{G}\supset\mathcal{G}_0$ and $0<\delta<\delta_0$.

A fact that we be used in this paper is that, $\mathrm{Bott}(h,v)|_{\mathcal{P}}$ *defines the same map* $\mathrm{Bott}(h,v)$ *as* $\mathrm{Bott}(h,v)|_{\mathcal{P}_0}$ *defines, if*

$$\|h(a)v-vh(a)\|<\delta\ \text{ for all }\ a\in\mathcal{G},$$

when $K_i(A)$ *is finitely generated.* In what follows, in the case that $K_i(A)$ is finitely generated, whenever we write $\mathrm{Bott}(h,v)$, we always assume that δ is smaller than δ_0 and $\mathcal{G}$ is larger than $\mathcal{G}_0$ so that $\mathrm{Bott}(h,v)$ is well-defined (see 2.10 of [29] for more details).

2.9. In the case that $A=C(S^1)$, there is a concrete way to visualize $\mathrm{bott}_1(h,v)$. It is perhaps helpful to describe it here. The map $\mathrm{bott}_1(h,v)$ is determined by $\mathrm{bott}_1(h,v)([z])$, where z is the identity map on the unit circle.

Denote $u=h(z)$ and define

$$f(e^{2\pi it})=\begin{cases}1-2t, & \text{if } 0\leq t\leq 1/2,\\ -1+2t, & \text{if } 1/2<t\leq 1,\end{cases}$$

$$g(e^{2\pi it})=\begin{cases}(f(e^{2\pi it})-f(e^{2\pi it})^2)^{1/2} & \text{if } 0\leq t\leq 1/2,\\ 0, & \text{if } 1/2<t\leq 1\ \text{ and}\end{cases}$$

$$h(e^{2\pi it})=\begin{cases}0, & \text{if } 0\leq t\leq 1/2,\\ (f(e^{2\pi it})-f(e^{2\pi it})^2)^{1/2}, & \text{if } 1/2<t\leq 1,\end{cases}$$

These are non-negative continuous functions defined on the unit circle. Suppose that $uv = vu$. Define

$$\mathtt{b}(u,v) = \begin{pmatrix} f(v) & g(v)+h(v)u^* \\ g(v)+uh(v) & 1-f(v) \end{pmatrix} \tag{e 2.8}$$

Then $\mathtt{b}(u,v)$ is a projection. There is $\delta_0 > 0$ (independent of unitaries u, v and A) such that if $\|[u,v]\| < \delta_0$, the spectrum of the positive element $\mathtt{p}(u,v)$ has a gap at $1/2$. The bott element of u and v is an element in $K_0(A)$ as defined in [11] and [12] which may be represented by

$$\mathrm{bott}_1(u,v) = [\chi_{[1/2,\infty)}(\mathtt{b}(u,v))] - [\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}]. \tag{e 2.9}$$

Note that $\chi_{[1/2,\infty)}$ is a continuous function on $\mathrm{sp}(\mathtt{b}(u,v))$. Suppose that $\mathrm{sp}(\mathtt{b}(u,v)) \subset (-\infty, a] \cup [1-a, \infty)$ for some $0 < a < 1/2$. Then $\chi_{[1/2,\infty)}$ can be replaced by any other positive continuous function F for which $F(t) = 0$ if $t \le a$ and $F(t) = 1$ if $t \ge 1/2$.

Definition 2.10. Let A and C be two unital C^*-algebras. Let $N : C_+ \setminus \{0\} \to \mathbb{N}$ and $K : C_+ \setminus \{0\} \to \mathbb{R}_+ \setminus \{0\}$ be two maps. Define $T = N \times K : C_+ \setminus \{0\} \to \mathbb{N} \times \mathbb{R}_+ \setminus \{0\}$ by $T(c) = (N(c), K(c))$ for $c \in C_+ \setminus \{0\}$. Let $L : C \to A$ be a unital positive linear map. We say L is T-full if for any $c \in C_+ \setminus \{0\}$, there are $x_1, x_2, ..., x_{N(c)} \in A$ with $\|x_i\| \le K(c)$ such that

$$\sum_{i=1}^{N(C)} x_i^* L(c) x_i = 1_A.$$

Let $\mathcal{H} \subset C_+ \setminus \{0\}$. We say that L is T-$\mathcal{H}$-full if

$$\sum_{i=1}^{N(c)} x_i^* L(c) x_i = 1_A$$

for all $c \in \mathcal{H}$.

Definition 2.11. Denote by $\mathcal{I}$ the class of unital C^*-algebras with the form $\bigoplus_{i=1}^m C(X_i, M_{n(i)})$, where $X_i = [0,1]$ or X_i is one point.

Definition 2.12. Let $k \ge 0$ be an integer. Denote by $\mathcal{I}_k$ the class of all C^*-algebras B with the form $B = PM_m(C(X))P$, where X is a finite CW complex with dimension no more than k, P is a projection in $M_m(C(X))$.

Recall that a unital simple C^*-algebra A is said to have tracial rank no more than k (write $TR(A) \le k$) if the following holds: For any $\epsilon > 0$, any positive element $a \in A_+ \setminus \{0\}$ and any finite subset $\mathcal{F} \subset A$, there exists a non-zero projection $p \in A$ and a C^*-subalgebra $B \in \mathcal{I}_k$ with $1_B = p$ such that

(1) $\|xp - px\| < \epsilon$ for all $x \in \mathcal{F}$;

(2) $pxp \in_\epsilon B$ for all $x \in \mathcal{F}$ and

(3) $1 - p$ is von Neumann equivalent to a projection in $\overline{aAa}$.

If $TR(A) \le k$ and $TR(A) \ne k-1$, we say A has tracial rank k and write $TR(A) \le k$. It has been shown that if $TR(A) = 1$, then, in the above definition, one can replace B by a C^*-algebra in $\mathcal{I}$ (see [19]). All unital simple AH-algebra with slow dimension growth and real rank zero have tracial rank zero (see [8] and also [22]) and all unital simple AH-algebras with no dimension growth have tracial rank no more than one (see [13], or, Theorem 2.5 of [28]). Note that all AH-algebras satisfy the Universal Coefficient Theorem. There are unital separable simple C^*-algebra A with $TR(A) = 0$ (and $TR(A) = 1$) which are not amenable.

3 Unitary groups

Lemma 3.1. *Let $H > 0$ be a positive number and let $N \geq 2$ be an integer. Then, for any unital C^*-algebra A, any projection $e \in A$ and any $u \in U_0(eAe)$ with $\mathrm{cel}_{eAe}(u) < H$,*

$$\mathrm{dist}(\overline{u + (1-e)}, \bar{1}) < H/N, \tag{e 3.10}$$

if there are mutually orthogonal and mutually equivalent projections $e_1, e_2, ..., e_{2N} \in (1-e)A(1-e)$ such that e_1 is also equivalent to e.

Proof. Since $\mathrm{cel}_{eAe}(u) < H$, there are unitaries $u_0, u_1, ..., u_N \in eAe$ such that

$$u_0 = u, \;\; u_N = 1 \;\text{ and }\; \|u_i - u_{i-1}\| < H/N, \;\; i = 1, 2, ..., N. \tag{e 3.11}$$

We will use the fact that

$$\begin{pmatrix} v & 0 \\ 0 & v^* \end{pmatrix} = \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} v^* & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In particular, $\begin{pmatrix} v & 0 \\ 0 & v^* \end{pmatrix}$ is a commutator. Note that

$$\|(u \oplus u_1^* \oplus u_1 \oplus u_2^* \oplus \cdots \oplus u_N^* \oplus u_N) - (u \oplus u^* \oplus u_1 \oplus u_1^* \cdots \oplus u_{N-1}^* \oplus u_N)\| < H/N. \tag{e 3.12}$$

Since $u_N = 1$, $u \oplus u^* \oplus u_1 \oplus u_1^* \cdots \oplus u_{N-1}^* \oplus u_N$ is a commutator.

Now we write

$$u \oplus e_1 \cdots \oplus e_{2N} = (u \oplus u_1^* \oplus u_1 \oplus \cdots \oplus u_N^* \oplus u_N)(e \oplus u_1 \oplus u_1^* \oplus \cdots \oplus u_N \oplus u_N^*).$$

We obtain $z \in CU((e + \sum_{i=1}^{2N} e_i)A(e + \sum_{i=1}^{2N} e_i))$ such that

$$\|u \oplus e_1 \cdots \oplus e_{2N} - z\| < H/N.$$

It follows that

$$\mathrm{dist}(\overline{u + (1-e)}, \bar{1}) < H/N.$$

∎

Definition 3.2. Let $C = PM_k(C(X))P$, where X is a compact metric space and $P \in M_k(C(X))$ is a projection. Let $u \in U(C)$. Recall (see [40]) that

$$D_c(u) = \inf\{\|a\| : a \in C_{s.a.} \;\text{ such that }\; \det(\exp(ia) \cdot u)(x) = 1 \;\text{ for all }\; x \in X\}.$$

If no self-adjoint element $a \in A_{s.a.}$ exists for which $\det(\exp(ia) \cdot u)(x) = 1$ for all $x \in X$, define $D_c(u) = \infty$.

Lemma 3.3. *Let $K \geq 1$ be an integer. Let A be a unital simple C^*-algebra with $TR(A) \leq 1$, let $e \in A$ be a projection and let $u \in U_0(eAe)$. Suppose that $w = u + (1-e)$ and suppose $\eta > 0$. Suppose also that*

$$[1-e] \leq K[e] \;\text{ in }\; K_0(A) \;\text{ and }\; \mathrm{dist}(\bar{w}, \bar{1}) < \eta. \tag{e 3.13}$$

Then, if $\eta < 2$,

$$\mathrm{cel}_{eAe}(u) < (\frac{K\pi}{2} + 1/16)\eta + 8\pi \;\text{ and }\; \mathrm{dist}(\bar{u}, \bar{e}) < (K + 1/8)\eta,$$

and if $\eta = 2$,

$$\mathrm{cel}_{eAe}(u) < \frac{K\pi}{2}\mathrm{cel}(w) + 1/16 + 8\pi.$$

Proof. We assume that (e 3.13) holds. Note that $\eta \le 2$. Put $L = \mathrm{cel}(w)$.

We first consider the case that $\eta < 2$. There is a projection $e' \in M_2(A)$ such that

$$[(1-e)+e'] = K[e].$$

To simplify notation, by replacing A by $(1_A+e')M_2(A)(1_A+e')$ and w by $w+e'$, without loss of generality, we may now assume that

$$[1-e] = K[e] \text{ and } \mathrm{dist}(\bar{w}, \bar{1}) < \eta. \tag{e 3.14}$$

There is $R_1 > 1$ such that $\max\{L/R_1, 2/R_1, \eta\pi/R_1\} < \min\{\eta/64, 1/16\pi\}$.

For any $\frac{\eta}{32K(K+1)\pi} > \epsilon > 0$ with $\epsilon + \eta < 2$, since $TR(A) \le 1$, there exists a projection $p \in A$ and a C^*-subalgebra $D \in \mathcal{I}$ with $1_D = p$ such that

(1) $\|[p, x]\| < \epsilon$ for $x \in \{u, w, e, (1-e)\}$,

(2) $pwp, pup, pep, p(1-e)p \in_\epsilon D$,

(3) there is a projection $q \in D$ and a unitary $z_1 \in qDq$ such that $\|q - pep\| < \epsilon$, $\|z_1 - quq\| < \epsilon$, $\|z_1 \oplus (p-q) - pwp\| < \epsilon$ and $\|z_1 \oplus (p-q) - c_1\| < \epsilon + \eta$;

(4) there is a projection $q_0 \in (1-p)A(1-p)$ and a unitary $z_0 \in q_0Aq_0$ such that $\|q_0 - (1-p)e(1-p)\| < \epsilon$, $\|z_0 - (1-p)u(1-p)\| < \epsilon$, $\|z_0 \oplus (1-p-q_0) - (1-p)w(1-p)\| < \epsilon$, $\|z_0 \oplus (1-p-q_0) - c_0\| < \epsilon + \eta$,

(5) $[p-q] = K[q]$ in $K_0(D)$, $[(1-p)-q_0] = K[q_0]$ in $K_0(A)$;

(6) $2(K+1)R_1[1-p] < [p]$ in $K_0(A)$;

(7) $\mathrm{cel}_{(1-p)A(1-p)}(z_0 \oplus (1-p-q_0)) \le L + \epsilon$,

where $c_1 \in CU(D)$ and $c_0 \in CU((1-p)A(1-p))$.

Note that $D_D(c_1) = 0$ (see 3.2). Since $\epsilon + \eta < 2$, there is $h \in D_{s.a}$ with $\|h\| \le 2\arcsin(\frac{\epsilon+\eta}{2})$ such that (by (3) above)

$$(z_1 \oplus (p-q))\exp(ih) = c_1. \tag{e 3.15}$$

It follows that

$$D_D((z_1 \oplus (p-q))\exp(ih)) = 0. \tag{e 3.16}$$

By (5) above and applying 3.3 of [40], one obtains that

$$|D_{qDq}(z_1)| \le K2\arcsin(\frac{\epsilon+\eta}{2}). \tag{e 3.17}$$

If $2K\arcsin(\frac{\epsilon+\eta}{2}) \ge \pi$, then

$$2K(\frac{\epsilon+\eta}{2})\frac{\pi}{2} \ge \pi.$$

It follows that

$$K(\epsilon+\eta) \ge 2 \ge \mathrm{dist}(\overline{z_1}, \overline{q}). \tag{e 3.18}$$

Since those unitaries in D with $\det(u) = 1$ (for all points) are in $CU(D)$ (see, for example, 3.5 of [9]), from (e 3.17), one computes that, when $2K\arcsin(\frac{\epsilon+\eta}{2}) < \pi$,

$$\mathrm{dist}(\overline{z_1}, \overline{q}) < 2\sin(K\arcsin(\frac{\epsilon+\eta}{2})) \le K(\epsilon+\eta). \tag{e 3.19}$$

By combining both (e 3.18) and (e 3.19), one obtains that

$$\mathrm{dist}(\overline{z_1}, \overline{q}) \le K(\epsilon+\eta) \le K\eta + \frac{\eta}{32(K+1)\pi}. \tag{e 3.20}$$

By (e 3.17), it follows from 3.4 of [40] that

$$\mathrm{cel}_{qDq}(z_1) \le 2K \arcsin\frac{\epsilon+\eta}{2} + 6\pi \le K(\epsilon+\eta)\frac{\pi}{2} + 6\pi \le (K\frac{\pi}{2} + \frac{1}{64(K+1)})\eta + 6\pi. \tag{e 3.21}$$

By (5) and (6) above,

$$(K+1)[q] = [p-q] + [q] = [p] > 2(K+1)R_1[1-p].$$

Since $K_0(A)$ is weakly unperforated, one has

$$2R_1[1-p] < [q]. \tag{e 3.22}$$

There is a unitary $v \in A$ such that

$$v^*(1-p-q_0)v \le q. \tag{e 3.23}$$

Put $v_1 = q_0 \oplus (1-p-q_0)v$. Then

$$v_1^*(z_0 \oplus (1-p-q_0))v_1 = z_0 \oplus v^*(1-p-q_0)v. \tag{e 3.24}$$

Note that

$$\|(z_0 \oplus v^*(1-p-q_0)v)v_1^*c_0^*v_1 - q_0 \oplus v^*(1-p-q_0)v\| < \epsilon + \eta. \tag{e 3.25}$$

Moreover, by (7) above,

$$\mathrm{cel}(z_0 \oplus v^*(1-p-q_0)v) \le L + \epsilon, \tag{e 3.26}$$

It follows from (e 3.22) and Lemma 6.4 of [28] that

$$\mathrm{cel}_{(q_0+q)A(q_0+q)}(z_0 \oplus q) \le 2\pi + (L+\epsilon)/R_1. \tag{e 3.27}$$

Therefore, combining (e 3.21),

$$\mathrm{cel}_{(q_0+q)A(q_0+q)}(z_0 + z_1) \le 2\pi + (L+\epsilon)/R_1 + (K\frac{\pi}{2} + \frac{1}{64(K+1)})\eta + 6\pi. \tag{e 3.28}$$

By (e 3.26), (e 3.22) and 3.1, in $U_0((q_0+q)A(q_0+q))/CU((q_0+q)A(q_0+q))$,

$$\mathrm{dist}(\overline{z_0+q}, \overline{q_0+q}) < \frac{(L+\epsilon)}{R_1}. \tag{e 3.29}$$

Therefore, by (e 3.19) and (e 3.29),

$$\mathrm{dist}(\overline{z_0 \oplus z_1}, \overline{q_0+q}) < \frac{(L+\epsilon)}{R_1} + K\eta + \frac{\eta}{32(K+1)\pi} < (K+1/16)\eta. \tag{e 3.30}$$

We note that

$$\|e - (q_0+q)\| < 2\epsilon \text{ and } \|u - (z_0+z_1)\| < 2\epsilon. \tag{e 3.31}$$

It follows that

$$\mathrm{dist}(\bar{u}, \bar{e}) < 4\epsilon + (K + 1/16)\eta < (K + 1/8)\eta. \tag{e 3.32}$$

Similarly, by (e 3.28),

$$\begin{aligned} \mathrm{cel}_{eAe}(u) &\le 4\epsilon\pi + 2\pi + (L+\epsilon)/R_1 + (K\frac{\pi}{2} + \frac{1}{64(K+1)})\eta + 6\pi & \text{(e 3.33)} \\ &< (K\frac{\pi}{2} + 1/16)\eta + 8\pi. & \text{(e 3.34)} \end{aligned}$$

This proves the case that $\eta < 2$.

Now suppose that $\eta = 2$. Define $R = [\mathrm{cel}(w) + 1]$. Note that $\frac{\mathrm{cel}(w)}{R} < 1$. There is a projection $e' \in M_{R+1}(A)$ such that

$$[(1-e) + e'] = (K + RK)[e].$$

It follows from 3.1 that

$$\mathrm{dist}(\overline{w \oplus e'}, \overline{1_A + e'}) < \frac{\mathrm{cel}(w)}{R+1}. \tag{e 3.35}$$

Put $K_1 = K(R+1)$. To simplify notation, without loss of generality, we may now assume that

$$[1-e] = K_1[e] \text{ and } \mathrm{dist}(\bar{w}, \bar{1}) < \frac{\mathrm{cel}(w)}{R+1}. \tag{e 3.36}$$

It follows from the first part of the lemma that

$$\begin{aligned} \mathrm{cel}_{eAe}(u) &< (\frac{K_1\pi}{2} + \frac{1}{16})\frac{\mathrm{cel}(w)}{R+1} + 8\pi & \text{(e 3.37)} \\ &\le \frac{K\pi \mathrm{cel}(w)}{2} + \frac{1}{16} + 8\pi. & \text{(e 3.38)} \end{aligned}$$

□

Theorem 3.4. *Let A be a unital simple C^*-algebra with $TR(A) \le 1$ and let $e \in A$ be a non-zero projection. Then the map $u \mapsto u + (1-e)$ induces an isomorphism j from $U(eAe)/CU(eAe)$ onto $U(A)/CU(A)$.*

Proof. It was shown in Theorem 6.7 of [28] that j is a surjective homomorphism. So it remains to show that it is also injective. To do this, fix a unitary $u \in eAe$ so that $\bar{u} \in \ker j$. We will show that $u \in CU(eAe)$.

There is an integer $K \ge 1$ such that

$$K[e] \ge [1-e] \text{ in } K_0(A).$$

Let $1 > \epsilon > 0$. Put $v = u + (1-e)$. Since $\bar{u} \in \ker j$, $v \in CU(A)$. In particular,

$$\mathrm{dist}(\bar{v}, \bar{1}) < \epsilon/(K\pi/2 + 1).$$

It follows from Lemma 3.3 that

$$\mathrm{dist}(\bar{u}, \bar{e}) < (\frac{K\pi}{2} + 1/16)(\epsilon/(K\pi/2+1)) < \epsilon.$$

It then follows that

$$u \in CU(eAe).$$

□

Corollary 3.5. *Let A be a unital simple C^*-algebra with $TR(A) \le 1$. Then the map $j : a \to \mathrm{diag}(a, \overbrace{1, 1, .., 1}^{m})$ from A to $M_n(A)$ induces an isomorphism from $U(A)/CU(A)$ onto $U(M_n(A))/CU(M_n(A))$ for any integer $n \ge 1$.*

4 Full spectrum

One should compare the following with Theorem 3.1 of [42].

Lemma 4.1. *Let X be a path connected finite CW complex, let $C = C(X)$ and let $A = C([0,1], M_n)$ for some integer $n \geq 1$. For any unital homomorphism $\phi : C \to A$, any finite subset $\mathcal{F} \subset C$ and any $\epsilon > 0$, there exists a unital homomorphism $\psi : C \to B$ such that*

$$\|\phi(c) - \psi(c)\| < \epsilon \text{ for all } c \in \mathcal{F} \text{ and} \tag{e 4.39}$$

$$\psi(f)(t) = W(t)^* \begin{pmatrix} f(s_1(t)) & & \\ & \ddots & \\ & & f(s_n((t))) \end{pmatrix} W(t), \tag{e 4.40}$$

where $W \in U(A)$, $s_j \in C([0,1], X)$, $j = 1, 2, ..., n$, and $t \in [0,1]$.

Proof. To simplify the notation, without loss of generality, we may assume that $\mathcal{F}$ is in the unit ball of C. Since X is also locally path connected, choose $\delta_1 > 0$ such that, for any point $x \in X$, $B(x, \delta_1)$ is path connected. Put $d = 2\pi/n$. Let $\delta_2 > 0$ (in place of δ) be as required by Lemma 2.6.11 of [21] for $\epsilon/2$.

We will also apply Corollary 2.3 of [42]. By Corollary 2.3 of [42], there exists a finite subset $\mathcal{H}$ of positive functions in $C(X)$ and $\delta_3 > 0$ satisfying the following: For any pair of points $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$, if $\{h(x_i)\}_{i=1}^n$ and $\{h(y_i)\}_{i=1}^n$ can be paired to within δ_3 one by one, in increasing order, counting multiplicity, for all $h \in \mathcal{H}$, then $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$ can be paired to within $\delta_3/2$, one by one.

Put $\epsilon_1 = \min\{\epsilon/16, \delta_1/16, \delta_2/4, \delta_3/4\}$. There exists $\eta > 0$ such that

$$|f(t) - f(t')| < \epsilon_1/2 \text{ for all } f \in \phi(\mathcal{F} \cup \mathcal{H}) \tag{e 4.41}$$

provided that $|t - t'| < \eta$. Choose a partition of the interval:

$$0 = t_0 < t_1 < \cdots < t_N = 1$$

such that $|t_i - t_{i-1}| < \eta$, $i = 1, 2, ..., N$. Then

$$\|\phi(f)(t_i) - \phi(f)(t_{i-1})\| < \epsilon_1 \text{ for all } f \in \mathcal{F} \cup \mathcal{H}, \tag{e 4.42}$$

$i = 1, 2, ..., N$. There are unitaries $U_i \in M_n$ and $\{x_{i,j}\}_{j=1}^n$, $i = 0, 1, 2, ..., N$, such that

$$\phi(f)(t_i) = U_i^* \begin{pmatrix} f(x_{i,1}) & & \\ & \ddots & \\ & & f(x_{i,n}) \end{pmatrix} U_i. \tag{e 4.43}$$

By the Weyl spectral variation inequality (see [1]), the eigenvalues of $\{h(x_{i,j})\}_{j=1}^n$ and $\{h(x_{i-1,j})\}_{j=1}^n$ can be paired to within δ_3, one by one, counting multiplicity, in decreasing order. It follows from Corollary 2.3 of [42] that $\{x_{i,j}\}_{j=1}^n$ and $\{x_{i-1,j}\}_{j=1}^n$ can be paired within $\delta_3/2$. We may assume that,

$$\mathrm{dist}(x_{i,\sigma_i(j)}, x_{i-1,j}) < \delta_3/2, \tag{e 4.44}$$

where $\sigma_i : \{1, 2, ..., n\} \to \{1, 2, ..., n\}$ is a permutation. By the choice of δ_3, there is continuous path $\{x_{i-1,j}(t) : t \in [t_{i-1}, (t_i + t_{i-1})/2]\} \subset B(x_{i-1}, \delta_3/2)$ such that

$$x_{i-1,j}(t_{i-1}) = x_{i-1,j} \text{ and } x_{i-1,j}((t_{i-1} + t_i)/2) = x_{i,\sigma_i(j)}, \tag{e 4.45}$$

$j = 1, 2, ..., n$. Put

$$\psi(f)(t) = U_{i-1}^* \begin{pmatrix} f(x_{i,1}(t)) & & \\ & \ddots & \\ & & f(x_{i,n}(t)) \end{pmatrix} U_{i-1} \tag{e 4.46}$$

for $t \in [t_{i-1}, (t_{i-1}+t_i)/2]$ and for $f \in C(X)$. In particular,

$$\psi(f)(\frac{t_{i-1}+t_i}{2}) = U_{i-1}^* \begin{pmatrix} f(x_{i,\sigma_i(1)}) & & \\ & \ddots & \\ & & f(x_{i,\sigma_i(n)}) \end{pmatrix} U_{i-1} \tag{e 4.47}$$

for $f \in C(X)$. Note that

$$\|\phi(f)(t_{i-1}) - \psi(f)(t)\| < \delta_2/4 \text{ and } \|\psi(f)(t) - \phi(f)(t_i)\| < \delta_2/4 + \epsilon_1/2 < \delta_2/2 \tag{e 4.48}$$

for all $f \in \mathcal{F}$ and $t \in [t_{i-1}, \frac{t_{i-1}+t_i}{2}]$. There exists a unitary $W_i \in M_n$ such that

$$W_i^* \psi(f)(\frac{t_{i-1}+t_i}{2}) W_i = \phi(f)(t_i) \tag{e 4.49}$$

for all $f \in C(X)$. It follows from (e 4.48) and (e 4.49) that

$$\|W_i \psi(f)(\frac{t_{i-1}+t_i}{2}) - \psi(f)(\frac{t_{i-1}+t_i}{2}) W_i\| < \delta_2 \tag{e 4.50}$$

for all $f \in \mathcal{F}$. By the choice of δ_2 and by applying Lemma 2.6.11 of [21], we obtain $h_i \in M_n$ such that $W_i = \exp(\sqrt{-1}h_i)$ and

$$\|h_i \psi(f)(\frac{t_{i-1}+t_i}{2}) - \psi(f)(\frac{t_{i-1}+t_i}{2}) h_i\| < \epsilon/4 \text{ and} \tag{e 4.51}$$

$$\|\exp(\sqrt{-1}th_i) \psi(f)(\frac{t_{i-1}+t_i}{2}) - \psi(f)(\frac{t_{i-1}+t_i}{2}) \exp(\sqrt{-1}th_i)\| < \epsilon/4 \tag{e 4.52}$$

for all $f \in \mathcal{F}$ and $t \in [0,1]$. From this we obtain a continuous path of unitaries $\{W_i(t) : t \in [\frac{t_{i-1}+t_i}{2}, t_i]\} \subset M_n$ such that

$$W_i(\frac{t_{i-1}+t_i}{2}) = 1, \;\; W_i(t_i) = W_i \text{ and} \tag{e 4.53}$$

$$\|W_i(t) \psi(f)(\frac{t_{i-1}+t_i}{2}) - \psi(f)(\frac{t_{i-1}+t_i}{2}) W_i(t)\| < \epsilon/4 \tag{e 4.54}$$

for all $f \in \mathcal{F}$ and $t \in [\frac{t_{i-1}+t_i}{2}, t_i]$. Define $\psi(f)(t) = W_i^*(t)\psi(\frac{t_{i-1}+t_i}{2})W_i(t)$ for $t \in [\frac{t_{i-1}+t_i}{2}, t_i]$, $i = 1, 2, ..., N$. Note that $\psi : C(X) \to A$. We conclude that

$$\|\phi(f) - \psi(f)\| < \epsilon \text{ for all } \mathcal{F}. \tag{e 4.55}$$

Define

$$U(t) = U_0 \text{ for } t \in [0, \frac{t_1}{2}), \;\; U(t) = U_0 W_1(t) \text{ for } t \in [\frac{t_1}{2}, t_2), \tag{e 4.56}$$

$$U(t) = U(t_i) \text{ for } t \in [t_i, \frac{t_i + t_{i+1}}{2}), \;\; U(t) = U(t_i) W_{i+1}(t) \text{ for } t \in [\frac{t_i + t_{i+1}}{2}, t_{i+1}], \tag{e 4.57}$$

$i = 1, 2, ..., N-1$ and define

$$s_j(t) = x_{0,j}(t) \text{ for } t \in [0, \frac{t_1}{2}), \;\; s_j(t) = s_j(\frac{t_1}{2}) \text{ for } t \in [\frac{t_1}{2}, t_2), \tag{e 4.58}$$

$$s_j(t) = x_{i,\sigma_i(j)}(t) \text{ for } t \in [t_i, \frac{t_i+t_{i+1}}{2}), \;\; s_j(t) = s_j(\frac{t_i+t_{i+1}}{2}) \text{ for } t \in [\frac{t_i+t_{i+1}}{2}, t_{i+1}], \tag{e 4.59}$$

$i=1,2,...,N-1$. Thus $U(t)\in A$ and, by (e 4.45), $s_j(t)\in C([0,1],X)$.

One then checks that ψ has the form:

$$\psi(f)=U(t)^*\begin{pmatrix} f(s_1(t)) & & \\ & \ddots & \\ & & f(s_n(t))\end{pmatrix}U(t) \tag{e 4.60}$$

for $f\in C(X)$. In fact, for $t\in[0,t_1]$, it is clear that (e 4.60) holds. Suppose that (e 4.60) holds for $t\in[0,t_i]$. Then, by (e 4.49), for $f\in C(X)$,

$$\psi(f)(t_i)=U(t_i)^*\begin{pmatrix} f(x_{i,\sigma_i(1)}) & & \\ & \ddots & \\ & & f(x_{i,\sigma_i(n)})\end{pmatrix}U(t_i)=U_i^*\begin{pmatrix} f(x_{i,1}) & & \\ & \ddots & \\ & & f(x_{i,n})\end{pmatrix}U_i. \tag{e 4.61}$$

Therefore, for $t\in[t_i,\frac{t_i+t_{i+1}}{2}]$,

$$\begin{aligned}\psi(f)(t) &= U_i^*\begin{pmatrix} f(x_{i,1}(t)) & & \\ & \ddots & \\ & & f(x_{i,n}(t))\end{pmatrix}U_i && \text{(e 4.62)}\\ &= U(t_i)^*\begin{pmatrix} f(x_{i,\sigma_i(1)}(t)) & & \\ & \ddots & \\ & & f(x_{i,\sigma_i(n)}(t))\end{pmatrix}U(t_i) && \text{(e 4.63)}\\ &= U(t)^*\begin{pmatrix} f(s_1(t)) & & \\ & \ddots & \\ & & f(s_n(t))\end{pmatrix}U(t). && \text{(e 4.64)}\end{aligned}$$

For $t\in[\frac{t_i+t_{i+1}}{2},t_{i+1}]$,

$$\begin{aligned}\psi(f)(t) &= W_{i+1}(t)^*\psi(\frac{t_i+t_{i+1}}{2})W_{i+1}(t) && \text{(e 4.65)}\\ &= W_{i+1}(t)^*U(t_i)^*\begin{pmatrix} f(s_1(\frac{t_i+t_{i+1}}{2})) & & \\ & \ddots & \\ & & f(s_n(\frac{t_i+t_{i+1}}{2}))\end{pmatrix}U(t_i)W_{i+1}(t) && \text{(e 4.66)}\\ &= U(t)^*\begin{pmatrix} f(s_1(t)) & & \\ & \ddots & \\ & & f(s_n(t))\end{pmatrix}U(t). && \text{(e 4.67)}\end{aligned}$$

This verifies (e 4.60).

□

Lemma 4.2. *Let X be a finite CW complex and let $A\in\mathcal{I}$. Suppose that $\phi: C(X)\otimes C(\mathbb{T})\to A$ is a unital homomorphism. Then, for any $\epsilon>0$ and any finite subset $\mathcal{F}\subset C(X)$, there exists a continuous path of unitaries $\{u(t):\, t\in[0,1]\}$ in A such that*

$$u(0)=\phi(1\otimes z),\;\; u(1)=1\;\text{ and }\;\|[\phi(f\otimes 1),\,u(t)]\|<\epsilon \tag{e 4.68}$$

for $f\in\mathcal{F}$ and $t\in[0,1]$.

Proof. It is clear that the general case can be reduced to the case that $A = C([0,1], M_n)$. Let $q_1, q_2, ..., q_m$ be projections of $C(X)$ corresponding to each path connected component of X. Since $\phi(q_i)A\phi(q_i) \cong C([0,1], M_{n_i})$ for some $1 \le n_i \le n$, $i = 1, 2, ...$, we may reduce the general case to the case that X is path connected and $A = C([0,1], M_n)$.

Note that we use z for the identity function on the unit circle.

For any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset C(X)$, by applying 4.1, one obtains a unital homomorphism $\psi : C(X) \otimes C(\mathbb{T}) \to A$ such that

$$\|\phi(g) - \psi(g)\| < \epsilon \text{ for all } g \in \{f \otimes 1 : f \in \mathcal{F}\} \cup \{1 \otimes z\} \text{ and} \tag{e 4.69}$$

$$\psi(f)(t) = U(t)^* \begin{pmatrix} f(s_1(t)) & & \\ & \ddots & \\ & & f(s_n(t)) \end{pmatrix} U(t), \tag{e 4.70}$$

for all $f \in C(X \times \mathbb{T})$, where $U(t) \in U(C([0,1], M_n))$, $s_j : [0,1] \to X \times \mathbb{T}$ is a continuous map, $j = 1, 2, ..., n$, and for all $t \in [0,1]$. There are continuous paths of unitaries $\{u_j(r) : r \in [0,1]\} \subset C([0,1])$ such that

$$u_j(0)(t) = (1 \otimes z)(s_j(t)), \;\; u_j(1) = 1, \;\; j = 1, 2, ..., n. \tag{e 4.71}$$

Define

$$u(r)(t) = U(t)^* \begin{pmatrix} u_j(r)(t) & & \\ & \ddots & \\ & & u_n(r)(t) \end{pmatrix} U(t). \tag{e 4.72}$$

Then

$$u(r)\psi(f \otimes 1) = \psi(f \otimes 1)u(r) \text{ for all } r \in [0,1].$$

It follows that

$$\|[\phi(f \otimes 1), u(r)]\| < \epsilon \text{ for all } r \in [0,1] \text{ and } \text{ for all } f \in \mathcal{F}.$$

□

Definition 4.3. Let X be a compact metric space. We say that X satisfies property (H) if the following holds:

For any $\epsilon > 0$, any finite subsets $\mathcal{F} \subset C(X)$ and any non-decreasing map $\Delta : (0,1) \to (0,1)$, there exists $\eta > 0$ (which depends on ϵ and $\mathcal{F}$ but not Δ), $\delta > 0$, a finite subset $\mathcal{G} \subset C(X)$ and a finite subset $\mathcal{P} \subset \underline{K}(C(X))$ satisfying the following:

Suppose that $\phi : C(X) \to C([0,1], M_n)$ is a unital δ-$\mathcal{G}$-multiplicative contractive completely positive linear map for which

$$\mu_{\tau \circ \phi}(O_a) \ge \Delta(a) \tag{e 4.73}$$

for any open ball O_a with radius $a \ge \eta$ and for all tracial states τ of $C([0,1], M_n)$, and

$$[\phi]|_{\mathcal{P}} = [\Phi]|_{\mathcal{P}}, \tag{e 4.74}$$

where Φ is a point-evaluation.

Then there exists a unital homomorphism $h : C(X) \to C([0,1], M_n)$ such that

$$\|\phi(f) - h(f)\| < \epsilon \tag{e 4.75}$$

for all $f \in \mathcal{F}$.

It is a restricted version of some relatively weakly semi-projectivity property. It has been shown in [34] that any k-dimensional torus has the property (H). So do those finite CW compleces X with torsion free $K_0(C(X))$ and torsion $K_1(C(X))$, any finite CW compleces with form $Y \times \mathbb{T}$ where Y is contractive and all one-dimensional finite CW compleces.

Theorem 4.4. *Let X be a finite CW complex for which $X \times \mathbb{T}$ has the property (H). Let $C = C(X)$ and let $\Delta : (0,1) \to (0,1)$ be a non-decreasing map. For any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset C$, there exists $\delta > 0$, $\eta > 0$ and there exists a finite subset $\mathcal{G} \subset C$ satisfying the following:*

Suppose that A is a unital simple C^*-algebra with $TR(A) \leq 1$, $\phi : C \to A$ is a unital homomorphism and $u \in A$ is a unitary and suppose that

$$\|[\phi(c),\, u]\| < \delta \text{ for all } c \in \mathcal{G} \text{ and } \mathrm{Bott}(\phi,\, u) = \{0\}. \tag{e 4.76}$$

Suppose also that there exists a unital contractive completely positive linear map $L : C \otimes C(\mathbb{T}) \to A$ such that (with z the identity function on the unit circle)

$$\|L(c \otimes 1) - \phi(c)\| < \delta,\ \ \|L(c \otimes z) - \phi(c)u\| < \delta \text{ for all } c \in \mathcal{G} \tag{e 4.77}$$

$$\text{and } \mu_{\tau \circ L}(O_a) \geq \Delta(a) \text{ for all } \tau \in T(A) \text{ and} \tag{e 4.78}$$

for all open balls O_a of $X \times \mathbb{T}$ with radius $1 > a \geq \eta$, where $\mu_{\tau \circ L}$ is the Borel probability measure defined by L. Then there exists a continuous path of unitaries $\{u(t) : t \in [0,1]\}$ in A such that

$$u(0) = u,\ \ u(1) = 1 \text{ and } \|[\phi(c),\, u(t)]\| < \epsilon \tag{e 4.79}$$

for all $c \in \mathcal{F}$ and for all $t \in [0,1]$.

Proof. Let $\Delta_1(a) = \Delta(a)/2$. Denote by $z \in C(\mathbb{T})$ the identity map on the unit circle. Let $B = C \otimes C(\mathbb{T}) = C(X \times \mathbb{T})$. Put $Y = X \times \mathbb{T}$. Without loss of generality, we may assume that $\mathcal{F}$ is in the unit ball of C. Let $\mathcal{F}_1 = \{c \otimes 1 : c \in \mathcal{F}\} \cup \{1 \otimes z\}$.

Let $\eta_1 > 0$ (in place of η), $\delta_1 > 0$ (in place of δ), $\mathcal{G}_1 \subset C(Y)$ (in place of $\mathcal{G}$) be a finite subset, $\mathcal{P}_1 \subset \underline{K}(C(Y))$ (in place of $\mathcal{P}$) and $\mathcal{U}_1 \subset U(C(Y))$ be as required by Theorem 10.8 of [34] corresponding to $\epsilon/16$ (in place of ϵ), $\mathcal{F}_1$ and Δ_1 (in place of Δ) above.

Without loss of generality, we may assume that

$$\mathcal{U}_1 = \{\zeta_1 \otimes 1, ..., \zeta_{K_1} \otimes 1,\ \ p_1 \otimes z \oplus (1 - p_1 \otimes 1), ..., p_{K_2} \otimes z \oplus (1 - p_{K_2} \otimes 1)\}, \tag{e 4.80}$$

where $\zeta_k \in U(C)$, $k = 1, 2, ..., K_1$ and $p_j \in C$ is a projection, $j = 1, 2, ..., K_2$. Denote $z_i = p_i \otimes z \oplus (1 - p_i \otimes 1)$, $i = 1, 2, .., K_2$. We may also assume that $\mathcal{U}_1 \subset U(M_k(C(Y)))$.

For any contractive completely positive linear map L' from $C(Y)$, we will also use L' for $L' \otimes \mathrm{id}_{M_k}$.

Fix a finite subset $\mathcal{G}_2 \subset C(Y)$ which contains $\mathcal{G}_1$. Choose a small $\delta_1' > 0$. We choose $\mathcal{G}_2$ so large and δ_1' so small that, for any δ_1'-$\mathcal{G}_2$-multiplicative map L' form $C(Y)$ to a unital C^*-algebra B', there are unitaries $w_1', w_2', ..., w_{K_1}'$ and $u_1', u_2', ..., u_{K_2}'$ in $M_k(B')$ such that

$$\|L'(\zeta_i) - w_i'\| < \delta_1/16 \text{ and } \|L'(z_j) - u_j'\| < \delta_1/16, \tag{e 4.81}$$

$i = 1, 2, ..., K_1$ and $j = 1, 2, ..., K_2$.

Let $\eta_2 > 0$ (in place of η), $\delta_2 > 0$ (in place of δ), $\mathcal{G}_3 \subset C(Y)$ (in place of $\mathcal{G}$) be required by 10.7 of [34] for $\min\{\delta_1/16, \delta_1'/16, \Delta_1(\eta_1)/16, \epsilon/16\}$ (in place of ϵ), $\mathcal{G}_1 \cup \mathcal{F}_1$ and Δ_1 (in place of Δ) above. We may assume that $\mathcal{G}_3 \supset \mathcal{G}_2 \cup \mathcal{G}_1 \cup \mathcal{F}_1$ and $\mathcal{G}_3$ is in the unit ball of $C(Y)$. Moreover, we may further assume that $\mathcal{G}_3 = \{c \otimes 1 : c \in \mathcal{F}_2\} \cup \{z \otimes 1, 1 \otimes z, 1 \otimes 1\}$ for some finite subset $\mathcal{F}_2$.

Suppose that $\mathcal{G} \subset A$ is a finite subset which contains at least $\mathcal{F}_2$. We may assume that $\delta_2 < \delta_1$. Let $\delta = \min\{\frac{\delta_1}{16}, \frac{\delta_1'}{16}, \frac{\Delta_1(\eta)}{4}\}$.

Let A be a unital simple C^*-algebra with $TR(A) \le 1$, let $\phi : C(X) \to A$ and $u \in U_0(A)$ be such that

$$\|[\phi(c), u]\| < \delta \text{ for all } c \in \mathcal{G} \text{ and } \mathrm{Bott}(\phi, u) = \{0\}. \tag{e 4.82}$$

We may assume that (e 4.77) holds for $\eta = \eta_1/2$. We also assume that there is a δ-$\mathcal{G}_3$-multiplicative contractive completely positive linear map $L : C(Y) \to A$ such that

$$\|L(f \otimes 1) - \phi(f)\| < \delta, \ \ \|L(1 \otimes z) - u\| < \delta \text{ and} \tag{e 4.83}$$
$$\mu_{\tau \circ L}(O_a) \ge 2\Delta_1(a) \text{ for all } a \ge \eta, \tag{e 4.84}$$

for all $\tau \in \mathrm{T}(A)$ and for all $f \in \mathcal{F}_2$. We will continue to use L for $L \otimes \mathrm{id}_{M_k}$.

By (e 4.81), one may also assume that there are unitaries $w_1, w_2, ..., w_{K_1}$ and unitaries $u_1, u_2, ..., u_{K_2}$ such that

$$\|L(\zeta_i \otimes 1) - w_i\| < \delta_1/16 \text{ and } \|L(z_j) - u_j\| < \delta_1/16, \tag{e 4.85}$$

$i = 1, 2, ..., K_1$ and $j = 1, 2, ..., K_2$.

We note that, by (e 4.76), $[u_i] = 0$ in $K_1(A)$. Put

$$H = \max\{\mathrm{cel}(u_i) : 1 \le i \le K_2\}.$$

Let $N \ge 1$ be an integer such that

$$\frac{\max\{1, \pi, H + \delta_1 + \delta\}}{N} < \delta/4 \text{ and } \frac{1}{N} < \Delta_1(\eta)/4. \tag{e 4.86}$$

For each i, there are self-adjoint elements $a_{i,1}, a_{i,2}, ..., a_{i,L(i)} \in A$ such that

$$u_i = \prod_{j=1}^{L(i)} \exp(\sqrt{-1}a_{i,j}) \text{ and } \sum_{i=1}^{L(i)} \|a_{i,j}\| \le H + \delta/64, \tag{e 4.87}$$

$i = 1, 2, ..., K_2$.

Put $\Lambda = \max\{L(i) : 1 \le i \le K_2\}$. Let $\epsilon_0 > 0$ such that if $\|p'a - ap'\| < \epsilon_0$ for any self-adjoint element a and projection p', $\|p' \exp(ia) - \exp(ia)p'\| < \delta_1/16\Lambda$.

By applying Corollary 10.7 of [34], we obtain mutually orthogonal projections $P_0, P_1, P_2 \in A$ with $P_0 + P_1 + P_2 = 1$ and a C^*-subalgebra $D = \bigoplus_{j=1}^{s} C(X_j, M_{r(j)})$, where $X_j = [0, 1]$ or X_j is a point, with $1_D = P_1$, a finite dimensional C^*-subalgebra $D_0 \subset A$ with $1_{D_0} = P_2$, a unital contractive completely positive linear map $L_0 : C(X) \to D_0$ and there exists a unital homomorphism $\Phi : C(Y) \to D$ such that

$$\|L(g) - (P_0 L(g) P_0 + L_0(g) + \Phi(g))\| < \min\{\frac{\delta_1}{16}, \frac{\delta_1'}{16}, \frac{\Delta_1(\eta_1)}{4}, \frac{\epsilon}{16}\} \text{ for all } g \in \mathcal{G}_2 \tag{e 4.88}$$
$$\text{and } (2N + 1)\tau(P_0 + P_2) < \tau(P_1) \text{ for all } \tau \in \mathrm{T}(A). \tag{e 4.89}$$

Moreover,

$$\|[x, P_0]\| < \min\{\epsilon_0, \delta_1/16, \delta_1'/16, \Delta_1(\eta_1)/4, \epsilon/16\} \tag{e 4.90}$$

for all $x \in \{: a_{i,j} 1 \le j \le L(i), 1 \le i \le K_2\} \cup \{L(\mathcal{G}_2)\}$.

There exists a unitary $u_j' \in M_k(P_0AP_0)$ and $u_j'' \in M_k(D_0)$ such that

$$\|u_j' - \overline{P_0}L(z_j)\overline{P_0}\| < \delta_1/16 \text{ and } \|u_j'' - L_0(z_j)\| < \delta_1/16 \tag{e 4.91}$$

where $\overline{P_0} = \mathrm{diag}(\overbrace{P_0, P_0, ..., P_0}^{k})$. It follows from (e 4.90) and (e 4.87) that $[u_j'] = 0$ in $K_1(A)$ and

$$\mathrm{cel}(u_j') \le H + \delta_1/16 + \delta/64, \;\; j = 1, 2, ..., K_2 \tag{e 4.92}$$

(in $M_k(P_0AP_0)$). Note also, since $u_j'' \in M_k(D_0)$,

$$\mathrm{cel}(u_j) \le \pi, \;\; j = 1, 2, ..., K_2.$$

By applying 4.2, there exists a continuous path of unitaries $\{v(t) : t \in [0, 1]\} \subset D$ such that

$$v(0) = \Phi(1 \otimes z), \;\; v(1) = P_1 \text{ and } \|[\Phi(f \otimes 1), v(t)]\| < \epsilon/4 \tag{e 4.93}$$

for all $t \in [0, 1]$. Define a contractive completely positive linear map $L_1 : C(Y) = C(X) \otimes C(\mathbb{T}) \to A$ by

$$L_1(f \otimes 1) = P_0L(f \otimes 1)P_0 + L_0(f \otimes 1) + \Phi(f \otimes 1) \text{ and} \tag{e 4.94}$$
$$L_1(1 \otimes g) = g(1) \cdot P_0 + g(1) \cdot P_2 + \Phi(1 \otimes g(z)). \tag{e 4.95}$$

for all $f \in C(X)$ and $g \in C(\mathbb{T})$. We compute (by choosing large $\mathcal{G}_2$) that

$$\mu_{\tau \circ L_1}(O_a) \ge \Delta_1(a) \text{ for all } a \ge \eta \text{ and} \tag{e 4.96}$$
$$|\tau \circ L_1(g) - \tau \circ L(g)| < \delta \text{ for all } g \in \mathcal{G}_1 \tag{e 4.97}$$

and (by the fact that $\mathrm{Bott}(\phi, u) = \{0\}$)

$$[L]|_{\mathcal{P}_1} = [L_1]|_{\mathcal{P}_1}. \tag{e 4.98}$$

We also have (by (e 4.83) and (e 4.88))

$$\mathrm{dist}(L^{\ddagger}(\zeta_i \otimes 1), L_1^{\ddagger}(\zeta_i \otimes 1)) < \delta_1/16, \;\; i = 1, 2, ..., K_1. \tag{e 4.99}$$

Moreover, for $j = 1, 2, ..., K_2$,

$$\mathrm{dist}(L^{\ddagger}(z_j), L_1^{\ddagger}(z_j)) < \delta_1/16 + \mathrm{dist}(\overline{(u_j' + u_j'' + \Phi(z_j))^*(\overline{P_0} + \overline{P_2} + \Phi(z_j))}, \bar{1}) \tag{e 4.100}$$
$$= \delta_1/16 + \mathrm{dist}(\overline{u_j' + u_j'' + \overline{P_1})^*}, \bar{1}) \tag{e 4.101}$$
$$< \delta_1/16 + \frac{\max\{\pi, H + \delta_1/16 + \delta\}/64}{N} \tag{e 4.102}$$
$$< \delta_1/16 + \delta/2 < \delta_1, \tag{e 4.103}$$

where the third inequality follows from (e 4.92) and 3.1.

From (e 4.96), (e 4.97), (e 4.98), and (e 4.102), by applying Theorem 10.8 of [34], one obtains a unitary $W \in A$ such that

$$\|\mathrm{ad}\, W \circ L_1(g) - L(g)\| < \epsilon/16 \text{ for all } g \in \{c \otimes 1 : c \in \mathcal{F} \otimes 1\} \cup \{1 \otimes z\}. \tag{e 4.104}$$

Define

$$u'(t) = W^*(P_0 \oplus v(t))W \;\; t \in [0, 1]. \tag{e 4.105}$$

Then $u'(0) = W^*(P_0 \oplus \Phi(1 \otimes z))W$ and $u'(1) = 1$. It follows from (e 4.93) and (e 4.104) that

$$\|[\phi(c),\, u'(t)]\| < \epsilon/2 \text{ for all } c \in \mathcal{F} \tag{e 4.106}$$

and for $t \in [0,1]$. Note that

$$\|u'(0) - u\| < \epsilon/8. \tag{e 4.107}$$

One then obtains a continuous path $\{u(t) : t \in [0,1]\} \subset A$ by connecting $u'(0)$ with u by a path with length no more than $\epsilon/2$. The theorem follows.

□

Corollary 4.5. *Let $C = C(X, M_n)$ where $X = [0,1]$ or $X = \mathbb{T}$ and $\Delta : (0,1) \to (0,1)$ be a non-decreasing map. For any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset C$, there exist $\delta > 0$, $\eta > 0$ and there exists a finite subset $\mathcal{G} \subset C$ satisfying the following:*

Suppose that A is a unital simple C^*-algebra with $TR(A) \leq 1$, $\phi : C \to A$ is a unital monomorphism and $u \in A$ is a unitary and suppose that

$$\|[\phi(c),\, u]\| < \delta \textit{ for all } c \in \mathcal{G}, \tag{e 4.108}$$

$$\mathrm{bott}_0(\phi, u) = \{0\} \textit{ and } \mathrm{bott}_1(\phi, u) = \{0\}. \tag{e 4.109}$$

Suppose also that there exists a unital contractive completely positive linear map $L : C \otimes C(\mathbb{T}) \to A$ such that (with z the identity function on the unit circle)

$$\|L(c \otimes 1) - \phi(c)\| < \delta,\ \ \|L(c \otimes z) - \phi(c)u\| < \delta \textit{ for all } c \in \mathcal{G} \text{ and } \mu_{\tau\circ L}(O_a) \geq \Delta(a) \tag{e 4.110}$$

for all open balls O_a of $[0,1] \times \mathbb{T}$ with radius $1 > a \geq \eta$, where $\mu_{\tau\circ L}$ is the Borel probability measure defined by restricting L on the center of $C \otimes C(\mathbb{T})$. Then there exists a continuous path of unitaries $\{u(t) : t \in [0,1]\}$ such that

$$u(0) = u,\ \ u(1) = 1 \text{ and } \|[\phi(c),\, u(t)]\| < \epsilon \tag{e 4.111}$$

for all $c \in \mathcal{F}$ and for all $t \in [0,1]$.

Corollary 4.6. *Let $C = C([0,1], M_n)$ and let $T = N \times K : (C \otimes C(\mathbb{T}))_+ \setminus \{0\} \to \mathbb{N} \times \mathbb{R}_+ \setminus \{0\}$ be a map. For any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset C$, there exists $\delta > 0$, a finite subset $\mathcal{H} \subset (C \otimes C(\mathbb{T}))_+ \setminus \{0\}$ and there exists a finite subset $\mathcal{G} \subset C$ satisfying the following:*

Suppose that A is a unital simple C^*-algebra with $TR(A) \leq 1$, $\phi : C \to A$ is a unital monomorphism and $u \in A$ is a unitary and suppose that

$$\|[\phi(c),\, u]\| \ < \ \delta \textit{ for all } c \in \mathcal{G} \text{ and} \tag{e 4.112}$$

$$\mathrm{bott}_0(\phi, u) \ = \ \{0\}. \tag{e 4.113}$$

Suppose also that there exists a unital contractive completely positive linear map $L : C \otimes C(\mathbb{T}) \to A$ which is T-$\mathcal{H}$-full such that (with z the identity function on the unit circle)

$$\|L(c \otimes 1) - \phi(c)\| < \delta \textit{ and } \|L(c \otimes z) - \phi(c)u\| < \delta \textit{ for all } c \in \mathcal{G}. \tag{e 4.114}$$

Then there exists a continuous path of unitaries $\{u(t) : t \in [0,1]\}$ in A such that

$$u(0) = u,\ \ u(1) = 1 \textit{ and } \|[\phi(c),\, u(t)]\| < \epsilon \tag{e 4.115}$$

for all $c \in \mathcal{F}$ and for all $t \in [0,1]$.

Proof. Fix $T = N \times K : \mathbb{N} \times \mathbb{R}_+ \setminus \{0\}$. Let $\Delta : (0,1) \to (0,1)$ be the non-decreasing map associated with T as in Proposition 11.2 of [34]. Let $\mathcal{G} \subset C$, $\delta > 0$ and $\eta > 0$ be as required by 4.5 for ϵ and $\mathcal{F}$ given and the above Δ.

It follows from 11.2 of [34] that there exists a finite subset $\mathcal{H} \subset (C \otimes C(\mathbb{T}))_+ \setminus \{0\}$ such that for any unital contractive completely positive linear map $L : C \otimes C(\mathbb{T}) \to A$ which is T-$\mathcal{H}$-full, one has that

$$\mu_{\tau \circ L}(O_a) \geq \Delta(a) \qquad \text{(e 4.116)}$$

for all open balls O_a of $X \times \mathbb{T}$ with radius $a \geq \eta$.

The corollary then follows immediately from 4.5.

□

The following is an easy but known fact.

Lemma 4.7. *Let $C = M_n$. Then, for any $\epsilon > 0$ and any finite subset $\mathcal{F}$, there exists $\delta > 0$ and a finite subset $\mathcal{G} \subset C$ satisfying the following: For any unital C^*-algebra A with $K_1(A) = U(A)/U_0(A)$ and any unital homomorphism $\phi : C \to A$ and any unitary $u \in A$ if*

$$\|[\phi(c),\, u]\| < \delta \ \textit{ and } \ \mathrm{bott}_0(\phi, u) = \{0\}, \qquad \text{(e 4.117)}$$

then there exists a continuous path of unitaries $\{u(t) : t \in [0,1]\} \subset A$ such that

$$u(0) = u, \ \ u(1) = 1 \ \textit{ and } \ \|[\phi(c),\, u(t)]\| < \epsilon \qquad \text{(e 4.118)}$$

for all $c \in \mathcal{F}$ and $t \in [0,1]$.

Proof. First consider the case that $\phi(c)$ commutes with u for all $c \in M_n$. Then one has a unital homomorphism $\Phi : M_n \otimes C(\mathbb{T}) \to A$ defined by $\Phi(c \otimes g) = \phi(c)g(u)$ for all $c \in C$ and $g \in C(\mathbb{T})$. Let $\{e_{i,j}\}$ be a matrix unit for M_n. Let $u_j = e_{j,j} \otimes z$, $j = 1, 2, ..., n$. The assumption $\mathrm{bott}_0(\phi, u) = \{0\}$ implies that $\Phi_{*1} = \{0\}$. It follows that $u_j \in U_0(A)$, $j = 1, 2, ..., n$. One then obtains a continuous path of unitaries $\{u(t) : t \in [0,1]\} \subset A$ such that

$$u(0) = u, \ \ u(1) = 1 \ \text{ and } \ \|[\phi(c), u(t)]\| = 0$$

for all $c \in C(\mathbb{T})$ and $t \in [0,1]$.

The general case follows from the fact that $C \otimes C(\mathbb{T})$ is weakly semi-projective.

□

Remark 4.8. Let X be a compact metric space and let A be a unital simple C^*-algebra. Suppose that $\phi : C(X) \to A$ is a unital injective completely positive linear map. Then it is easy to check (see 7.2 of [29], for example) that there exists a non-decreasing map $\Delta : (0,1) \to (0,1)$ such that

$$\mu_{\tau \circ \phi}(O_a) \geq \Delta(a)$$

for all $a \in (0,1)$ and for all $\tau \in \mathrm{T}(A)$.

5 Changing spectrum

Lemma 5.1. *Let $n \geq 64$ be an integer. Let $\epsilon > 0$ and $1/2 > \epsilon_1 > 0$. There exists $\frac{\epsilon}{2n} > \delta > 0$ and a finite subset $\mathcal{G} \subset D \cong M_n$ satisfying the following:*

Suppose that A is a unital C^*-algebra with $T(A) \neq \emptyset$, $D \subset A$ is a C^*-subalgebra with $1_D = 1_A$, suppose that $\mathcal{F} \subset A$ is a finite subset and suppose that $u \in U(A)$ such that

$$\begin{aligned} \|[f, x]\| &< \delta \text{ for all } f \in \mathcal{F} \text{ and } x \in \mathcal{G}, \text{ and} && \text{(e 5.119)} \\ \|[u, x]\| &< \delta \text{ for all } x \in \mathcal{G}. && \text{(e 5.120)} \end{aligned}$$

Then, there exists a unitary $v \in D$ and a continuous path of unitaries $\{w(t) : t \in [0,1]\} \subset D$ such that

$$\begin{aligned} &\|[u, w(t)]\| < n\delta < \epsilon, \ \ \|[f, w(t)]\| < n\delta < \epsilon/2 && \text{(e 5.121)} \\ &\text{for all } f \in \mathcal{F} \text{ and for all } t \in [0,1], && \text{(e 5.122)} \\ &w(0) = 1, \ \ w(1) = v \text{ and } \mu_{\tau\circ\imath}(I_a) \geq \frac{2}{3n^2} && \text{(e 5.123)} \end{aligned}$$

for all open arcs I_a of $\mathbb{T}$ with length $a \geq 4\pi/n$ and for all $\tau \in T(A)$, where $\imath : C(\mathbb{T}) \to A$ is defined by $\imath(f) = f(vu)$ for all $f \in C(\mathbb{T})$.

Moreover,

$$\mathrm{length}(\{w(t)\}) \leq \pi. \qquad \text{(e 5.124)}$$

If, in addition, $\pi > b_1 > b_2 > \cdots > b_m > 0$ and $1 = d_0 > d_1 > d_2 \geq \cdots > d_m > 0$ are given so that

$$\mu_{\tau\circ\imath_0}(I_{b_i}) \geq d_i \text{ for all } \tau \in T(A), \ \ i = 1, 2, ..., m, \qquad \text{(e 5.125)}$$

where $\imath_0 : C(\mathbb{T}) \to A$ is defined by $\imath_0(f) = f(u)$ for all $f \in C(\mathbb{T})$, then one also has that

$$\mu_{\tau\circ\imath}(I_{c_i}) \geq (1 - \epsilon_1)d_i \text{ for all } \tau \in T(A), \qquad \text{(e 5.126)}$$

where I_{b_i} and I_{c_i} are any open arcs with length b_i and c_i, respectively, and where $c_i = b_i + \epsilon_1$, $i = 1, 2, ..., m$.

Proof. Let

$$0 < \delta_0 < \min\{\frac{\epsilon_1 d_i}{16n^2} : 1 \leq i \leq m\}.$$

Let $\{e_{i,j}\}$ be a matrix unit for D and let $\mathcal{G} = \{e_{i,j}\}$. Define

$$v = \sum_{j=1}^{n} e^{2\sqrt{-1}j\pi/n} e_{j,j}. \qquad \text{(e 5.127)}$$

Let $f_1 \in C(\mathbb{T})$ with $f_1(t) = 1$ for $|t - e^{2\sqrt{-1}\pi/n}| < \pi/n$ and $f_1(t) = 0$ if $|t - e^{2\sqrt{-1}\pi/n}| \geq 2\pi/n$ and $1 \geq f_1(t) \geq 0$. Define $f_{j+1}(t) = f_1(e^{2\sqrt{-1}j\pi/n}t)$, $j = 1, 2, ..., n-1$. Note that

$$f_i(e^{2\sqrt{-1}j\pi/n}t) = f_{i+j}(t) \text{ for all } t \in \mathbb{T} \qquad \text{(e 5.128)}$$

where $i, j \in \mathbb{Z}/n\mathbb{Z}$.

Fix a finite subset $\mathcal{F}_0 \subset C(\mathbb{T})_+$ which contains f_i, $i = 1, 2, ..., n$.

Choose δ so small that the following hold:

(1) there exists a unitary $u_i \in e_{i,i}Ae_{i,i}$ such that $\|e^{2\sqrt{-1}i\pi/n}e_{i,i}ue_{i,i} - u_i\| < \delta_0^2/16n^2$, $i = 1, 2, ..., n$;

(2) $\|e_{i,j}f(u) - f(u)e_{i,j}\| < \delta_0^2/16n^2$ for all $f \in \mathcal{F}_0$,

(3) $\|e_{i,i}f(vu)-e_{i,i}f(e^{2\sqrt{-1}i\pi/n}u)\|<\delta_0^2/16n^2$ for all $f\in\mathcal{F}_0$ and

(4) $\|e_{i,j}^*f(u)e_{i,j}-e_{j,j}f(u)e_{j,j}\|<\delta_0^2/16n^2$ for all $f\in\mathcal{F}_0$.

Fix k. For each $\tau\in \mathrm{T}(A)$, by (2), (3) and (4) above, there is at least one i such that

$$\tau(e_{j,j}f_i(u))\ge 1/n^2-\delta_0^2/16n^2. \tag{e 5.129}$$

Choose j so that $k+j=i$ mod (n). Then,

$$\begin{aligned}\tau(f_k(vu)) &\ge \tau(e_{j,j}f_k(vu)) & \text{(e 5.130)}\\ &\ge \tau(e_{j,j}f_k(e^{2\sqrt{-1}j\pi/n}u)))-\frac{\delta_0^2}{16n^2} & \text{(e 5.131)}\\ &= \tau(e_{j,j}f_i(u))-\frac{\delta_0^2}{16n^2}\ge\frac{1}{n^2}-\frac{2\delta_0^2}{16n^2}. & \text{(e 5.132)}\end{aligned}$$

It follows that

$$\mu_{\tau\circ\imath}(B(e^{2\sqrt{-1}k\pi/n},\pi/n))\ge\frac{1}{n^2}-\frac{2\delta_0^2}{16n^2}\ \text{ for all }\ \tau\in\mathrm{T}(A) \tag{e 5.133}$$

and for $k=1,2,...,n$.

It is then easy to compute that

$$\mu_{\tau\circ\imath}(I_a)\ge 2/3n^2\ \text{ for all }\ \tau\in\mathrm{T}(A) \tag{e 5.134}$$

and for any open arc with length $a\ge 2(2\pi/n)=4\pi/n$.

Note that if $\|[x,\,e_{i,i}]\|<\delta$, then

$$\|[x,\sum_{i=1}^n\lambda_ie_{i,i}]\|<n\delta<\epsilon/2\ \text{ and }\ \|[u,\sum_{i=1}^n\lambda_ie_{i,i}\|<n\delta<\epsilon/2$$

for any $\lambda_i\in\mathbb{T}$. Thus, one obtains a continuous path $\{w(t):t\in[0,1]\}\subset D$ with length$(\{w(t)\})\le\pi$ and with $w(0)=1$ and $w(1)=v$ so that (e 5.121) holds.

Let $\{x_1,x_2,...,x_K\}$ be an $\epsilon_1/64$-dense set of $\mathbb{T}$. Let $I_{i,j}$ be an open arc with center x_j and length b_i, $j=1,2,...,K$ and $i=1,2,...,m$. For each j and i, there is a positive function $g_{j,i}\in C(\mathbb{T})_+$ with $0\le g_{j,i}\le 1$ and $g_{j,i}(t)=1$ if $|t-x_j|<d_i$ and $g_{j,i}(t)=0$ if $|t-x_j|\ge d_i+\epsilon_1/64$, $j=1,2,...,K$, $i=1,2,...,m$. Put $g_{i,j,k}(t)=g_{j,i}(e^{2\sqrt{-1}k\pi/n}\cdot t)$ for all $t\in\mathbb{T}$, $k=1,2,...,n$. Suppose that $\mathcal{F}_0$ contains all $g_{j,i}$ and $g_{j,i,k}$. We have, by (2), (3) and (4) above,

$$\tau(g_{j,i}(u)e_{l,l}),\ \tau(g_{j,i,k}(u)e_{l,l})\ge\frac{d_i}{n}-\delta^2/16n^2\ \text{ for all }\ \tau\in\mathrm{T}(A), \tag{e 5.135}$$

$l=1,2,...,n,\ j=1,2,...,K$ and $i=1,2,...,m$. Thus

$$\begin{aligned}\tau(e_{k,k}g_{j,i}(vu)) &\ge \tau(e_{k,k}g_{j,i}(e^{2\sqrt{-1}k\pi/n}u)-n\frac{\delta_0^2}{16n^2} & \text{(e 5.136)}\\ &\ge \frac{d_i}{n}-\frac{\delta_0^2}{8n}\ \text{ for all }\ \tau\in\mathrm{T}(A), & \text{(e 5.137)}\end{aligned}$$

$k=1,2,...,n,\ j=1,2,...,K$ and $i=1,2,...,m$. Therefore

$$\tau(g_{j,i}(vu))\ge d_i-\frac{\delta_0^2}{8n}\ge(1-\epsilon_1)d_i\ \text{ for all }\ \tau\in\mathrm{T}(A), \tag{e 5.138}$$

$j = 1, 2, ..., K$ and $i = 1, 2, ..., m$.

It follows that

$$\mu_{\tau\circ\imath}(I_{i,j}) \geq (1 - \epsilon_1)d_i \text{ for all } \tau \in \mathrm{T}(A), \tag{e 5.139}$$

$j = 1, 2, ..., K$ and $i = 1, 2, ..., m$. Since $\{x_1, x_2, ..., x_K\}$ is $\epsilon_1/64$-dense in $\mathbb{T}$, it follows that

$$\mu_{\tau\circ\imath}(I_{c_i}) \geq (1 - \epsilon_1)d_i \text{ for all } \tau \in \mathrm{T}(A), \quad i = 1, 2, ..., m. \tag{e 5.140}$$

$\square$

Remark 5.2. If the assumption that $\|[f, x]\| < \delta$ for all $f \in \mathcal{F}$ and for all $x \in \mathcal{G}$ is replaced by for all $x \in D$ with $\|x\| \leq 1$, then the conclusion can also be strengthened to $\|[f, w(t)]\| < \delta$ for all $f \in \mathcal{F}$ and $t \in [0, 1]$.

The proof of the following is similar to that of 5.1.

Lemma 5.3. *Let $n \geq 64$ be an integer. Let $\epsilon > 0$ and $1/2 > \epsilon_1 > 0$. There exists $\frac{\epsilon}{2n} > \delta > 0$ and a finite subset $\mathcal{G} \subset D \cong M_n$ satisfying the following:*

Suppose that X is a compact metric space, $\mathcal{F} \subset C(X)$ is a finite subset and $1 > b > 0$. Then there exists a finite subset $\mathcal{F}_1 \subset C(X)$ satisfying the following:

Suppose that A is a unital C^*-algebra with $T(A) \neq \emptyset$, $D \subset A$ is a C^*-subalgebra with $1_D = 1_A$, $\phi : C(X) \to A$ is a unital homomorphism and suppose that $u \in U(A)$ such that

$$\|[x, u]\| < \delta \text{ and } \|[x, \phi(f)]\| < \delta \text{ for all } x \in \mathcal{G} \text{ and } f \in \mathcal{F}_1. \tag{e 5.141}$$

Suppose also that, for some $\sigma > 0$,

$$\tau(\phi(f)) \geq \sigma \text{ for all } \tau \in T(A) \text{ and} \tag{e 5.142}$$

for all $f \in C(X)$ with $0 \leq f \leq 1$ whose support contains an open ball of X with radius b. Then, there exists a unitary $v \in D$ and a continuous path of unitaries $\{v(t) : t \in [0, 1]\} \subset D$ such that

$$\|[u, v(t)]\| < n\delta < \epsilon, \quad \|[\phi(f), v(t)]\| < n\delta < \epsilon \tag{e 5.143}$$

$$\text{for all } f \in \mathcal{F} \text{ and } t \in [0, 1], \tag{e 5.144}$$

$$v(0) = 1, \quad v(1) = v \text{ and} \tag{e 5.145}$$

$$\tau(\phi(f)g(vu)) \geq \frac{2\sigma}{3n^2} \text{ for all } \tau \in T(A) \tag{e 5.146}$$

for any pair of $f \in C(X)$ with $0 \leq f \leq 1$ whose support contains an open ball with radius $2b$ and $g \in C(\mathbb{T})$ with $0 \leq g \leq 1$ whose support contains an open arc of $\mathbb{T}$ with length at least $8\pi/n$.

Moreover,

$$\mathrm{length}(\{v(t)\}) \leq \pi. \tag{e 5.147}$$

If, in addition, $1 > b_1 > b_2 > \cdots > b_k > 0$, $1 > d_1 \geq d_2 \geq \cdots \geq d_k > 0$ are given and

$$\tau(\phi(f')g'(u)) \geq d_i \text{ for all } \tau \in T(A) \tag{e 5.148}$$

for any functions $f' \in C(X)$ with $0 \leq f' \leq 1$ whose support contains an open ball of X with radius $b_i/2$ and $g' \in C(\mathbb{T})$ with $0 \leq g' \leq 1$ whose support contains an arc with length b_i, then one also has that

$$\tau(\phi(f'')g''(vu)) \geq (1 - \epsilon_1)d_i \text{ for all } \tau \in T(A), \tag{e 5.149}$$

where $f'' \in C(X)$ with $0 \leq f'' \leq 1$ whose support contains an open ball of radius c_i and $g'' \in C(\mathbb{T})$ with $0 \leq g'' \leq 1$ whose support contains an arc with length $2c_i$ with $c_i = b_i + \epsilon_1$, $i = 1, 2, ..., k$.

Proof. Let $0 < \delta_0 = \min\{\frac{\epsilon_1 d_i}{16n^2} : i = 1, 2, ..., k\}$.

Let $\{e_{i,j}\}$ be a matrix unit for D and let $\mathcal{G} = \{e_{i,j}\}$. Define

$$v = \sum_{j=1}^{n} e^{2\sqrt{-1}j\pi/n} e_{j,j}. \tag{e 5.150}$$

Let $g_j \in C(\mathbb{T})$ with $g_j(t) = 1$ for $|t - e^{2\sqrt{-1}j\pi/n}| < \pi/n$ and $g_j(t) = 0$ if $|t - e^{2\sqrt{-1}j\pi/n}| \ge 2\pi/n$ and $1 \ge g_j(t) \ge 0$, $j = 1, 2, ..., n$. As in the proof 5.1, we may also assume that

$$g_i(e^{2\sqrt{-1}j\pi/n}t) = g_{i+j}(t) \text{ for all } t \in \mathbb{T} \tag{e 5.151}$$

where $i, j \in \mathbb{Z}/n\mathbb{Z}$.

Let $\{x_1, x_2, ..., x_m\}$ be a $b/2$-dense subset of X. Define $f_i \in C(X)$ with $f_i(x) = 1$ for $x \in B(x_i, b)$ and $f_i(x) = 0$ if $x \notin B(x_i, 2b)$ and $0 \le f_i \le 1$, $i = 1, 2, ..., m$.

Note that

$$\tau(\phi(f_i)) \ge \sigma \text{ for all } \tau \in \mathrm{T}(A), \;\; i = 1, 2, ..., m. \tag{e 5.152}$$

Fix a finite subset $\mathcal{F}_0 \subset C(\mathbb{T})$ which at least contains $\{g_1, g_2, ..., g_n\}$ and $\mathcal{F}_1 \subset C(X)$ which at least contains $\mathcal{F}$ and $\{f_1, f_2, ..., f_m\}$.

Choose δ so small that the following hold:

(1) there exists a unitary $u_i \in e_{i,i}Ae_{i,i}$ such that $\|e^{2\sqrt{-1}i\pi/n}e_{i,i}ue_{i,i} - u_i\| < \delta_0^2/16n^4$, $i = 1, 2, ..., n$;

(2) $\|e_{i,j}g(u) - g(u)e_{i,j}\| < \delta_0^2/16n^4$, $\|e_{i,j}\phi(f) - \phi(f)e_{i,j}\| < \delta_0^2/16n^4$, for $f \in \mathcal{F}_1$ and $g \in \mathcal{F}_0$, j, $k = 1, 2, ..., n$ and $s = 1, 2, ..., m$;

(3) $\|e_{i,i}g(vu) - e_{i,i}g(e^{2\sqrt{-1}i\pi/n}u)\| < \delta_0^2/16n^4$ for all $g \in \mathcal{F}_0$ and

(4) $\|e_{i,j}^*g(u)e_{i,j} - e_{j,j}g(u)e_{j,j}\| < \delta_0^2/16n^4$, $\|e_{i,j}^*\phi(f)e_{i,j} - e_{j,j}\phi(f)e_{j,j}\| < \delta_0^2/16n^4$ for all $f \in \mathcal{F}_1$ and $g \in \mathcal{F}_0$, $j, k = 1, 2, ..., n$ and $s = 1, 2, ..., m$.

It follows from (4) that, for any $k_0 \in \{1, 2, ..., m\}$,

$$\tau(\phi(f_{k_0})e_{j,j}) \ge \sigma/n - n\delta_0^2/16n^4. \tag{e 5.153}$$

Fix k_0 and k. For each $\tau \in \mathrm{T}(A)$, there is at least one i such that

$$\tau(\phi(f_{k_0})e_{j,j}g_i(u)) \ge \sigma/n^2 - \delta_0^2/16n^4. \tag{e 5.154}$$

Choose j so that $k + j = i \bmod (n)$. Then,

$$\begin{aligned} \tau(\phi(f_{k_0})g_k(vu)) &\ge \tau(\phi(f_{k_0})e_{j,j}g_k(e^{2\sqrt{-1}j\pi/n}u))) - \frac{\delta_0^2}{16n^4} && \text{(e 5.155)} \\ &= \tau(\phi(f_{k_0})e_{j,j}g_i(u)) - \frac{\delta_0^2}{16n^4} && \text{(e 5.156)} \\ &\ge \frac{\sigma}{n^2} - \frac{2\delta_0^2}{16n^4} \text{ for all } \tau \in \mathrm{T}(A). && \text{(e 5.157)} \end{aligned}$$

It is then easy to compute that

$$\tau(\phi(f)g(vu)) \ge \frac{2\sigma}{3n^2} \text{ for all } \tau \in \mathrm{T}(A) \tag{e 5.158}$$

and for any pair of $f \in C(X)$ with $0 \le f \le 1$ whose support contains an open ball with radius $2b$ and $g \in C(\mathbb{T})$ with $0 \le g \le 1$ whose support contains an open arc of length at least $8\pi/n$.

Note that if $\|[\phi(f),\, e_{i,i}]\| < \delta$, then

$$\|[\phi(f), \sum_{i=1}^n \lambda_i e_{i,i}]\| < n\delta < \epsilon$$

for any $\lambda_i \in \mathbb{T}$ and $f \in \mathcal{F}_1$. We then also require that $\delta < \epsilon/2n$. Thus, one obtains a continuous path $\{v(t) : t \in [0,1]\} \subset D$ with length$(\{v(t)\}) \le \pi$ and with $v(0) = 1$ and $v(1) = v$ so that the second part of (e 5.143) holds.

Now we consider the last part of the lemma. Note also that, if $f \in \mathcal{F}_1$ and $g \in \mathcal{F}_0$ with $0 \le f, g \le 1$,

$$\begin{aligned} \tau(\phi(f)g(vu)) &\ge \sum_{j=1}^n \tau(\phi(f)e_{j,j}g(vu)) - \frac{\delta_0^2}{16n^3} && \text{(e 5.159)} \\ &\ge \sum_{j=1}^n \tau(\phi(f)e_{j,j}g^{(j)}(vu)) - \frac{\delta_0^2}{16n^2} \ \text{ for all } \ \tau \in \mathrm{T}(A), && \text{(e 5.160)} \end{aligned}$$

where $g^{(j)}(t) = g(e^{2\sqrt{-1}j\pi/n} \cdot t)$ for $t \in \mathbb{T}$. If the support of f contains an open ball with radius $b_i/2$ and that of g contains open arcs with length at least b_i, so does that of $g^{(j)}$. So, if $\mathcal{F}_0$ and $\mathcal{F}_1$ are sufficiently large, by the assumptions of the last part of the lemma, as in the proof of 5.1, we have

$$\tau(\phi(f)g(vu)) \ge d_i - \frac{\delta_0^2}{16n^2} \ \text{ for all } \ \tau \in \mathrm{T}(A) \qquad \text{(e 5.161)}$$

for all $\tau \in \mathrm{T}(A)$. As in the proof of 5.1, this lemma follows when we choose $\mathcal{F}_0$ and $\mathcal{F}_1$ large enough to begin with.

$\square$

Lemma 5.4. *Let C be a unital separable simple C^*-algebra with $TR(C) \le 1$ and let $n \ge 1$ be an integer. For any $\epsilon > 0$, $\eta > 0$, any finite subset $\mathcal{F} \subset C$, there exists $\delta > 0$, a projection $p \in A$ and a C^*-subalgebra $D \cong M_n$ with $1_D = p$ such that*

$$\begin{aligned} \|[x,p]\| &< \epsilon \text{ for all } x \in \mathcal{F}; && \text{(e 5.162)} \\ \|[pxp, y]\| &< \epsilon \text{ for all } x \in \mathcal{F} \text{ and } y \in D \text{ with } \|y\| \le 1 \text{ and} && \text{(e 5.163)} \\ \tau(1-p) &< \eta \text{ for all } \tau \in T(C). && \text{(e 5.164)} \end{aligned}$$

Proof. Choose an integer $N \ge 1$ such that $1/N < \eta/2n$ and $N \ge 2n$. It follows from (the proof of) Theorem 5.4 of [28] that there is a projection $q \in C$ and there exists a C^*-subalgebra B of C with $1_B = q$ and $B \cong \oplus_{i=1}^L M_{K_i}$ with $K_i \ge N$ such that

$$\begin{aligned} &\|[x,q]\| < \eta/4 \ \text{ for all } \ x \in \mathcal{F}; && \text{(e 5.165)} \\ &\|[qxq,\, y]\| < \epsilon/4 \ \text{ for all } \ x \in \mathcal{F} \ \text{ and } \ y \in B \ \text{ with } \ \|y\| \le 1 \ \text{ and} && \text{(e 5.166)} \\ &\tau(1-q) < \eta/2n \ \text{ for all } \ \tau \in T(C). && \text{(e 5.167)} \end{aligned}$$

Write $K_i = k_i n + r_i$ with $k_i \ge 1$ and $0 \le r_i < n$ for some integers k_i and r_i, $i = 1, 2, ..., L$. Let $p \in B$ be a projection such that the rank of p is k_i in each summand M_{K_i} of B. Take $D_1 = pBp$. We have

$$\begin{aligned} &\|[x,p]\| < \epsilon/2 \ \text{ for all } \ x \in \mathcal{F}; && \text{(e 5.168)} \\ &\|[pxp,\, y]\| < \epsilon \ \text{ for all } \ x \in \mathcal{F}, y \in D_1 \ \text{ with } \ \|y\| \le 1 \ \text{ and} && \text{(e 5.169)} \\ &\tau(1-p) < \eta/2n + n/N < \eta/2n + \eta/2 < \eta \ \text{ for all } \ \tau \in T(C). && \text{(e 5.170)} \end{aligned}$$

Note that there is a unital C^*-subalgebra $D \subset D_1$ such that $D \cong M_n$.

$\square$

Lemma 5.5. *Let $n \geq 1$ be an integer with $n \geq 64$. Let $\epsilon > 0$ and $1/2 > \epsilon_1 > 0$. Suppose that A is a unital simple C^*-algebra with $TR(A) \leq 1$, suppose that $\mathcal{F} \subset A$ is a finite subset and suppose that $u \in U(A)$. Then, for any $\epsilon > 0$, there exists a unitary $v \in A$ and a continuous path of unitaries $\{w(t) : t \in [0,1]\} \subset A$ such that*

$$\|[x, w(t)]\| < \epsilon \text{ for all } f \in \mathcal{F} \text{ and } \text{ for all } t \in [0,1], \tag{e 5.171}$$
$$w(0) = 1, \;\; w(1) = v \;\text{ and} \tag{e 5.172}$$
$$\mu_{\tau\circ\imath}(I_a) \geq \frac{15}{24n^2} \tag{e 5.173}$$

for all open arcs I_a of $\mathbb{T}$ with length $a \geq 4\pi/n$ and for all $\tau \in T(A)$, where $\imath : C(\mathbb{T}) \to A$ is defined by $\imath(f) = f(vu)$. Moreover,

$$\text{length}(\{w(t)\}) \leq \pi. \tag{e 5.174}$$

If, in addition, $\pi > b_1 > b_2 > \cdots > b_m > 0$ and $1 = d_0 > d_1 > d_2 \geq \cdots > d_m > 0$ are given so that

$$\mu_{\tau\circ\imath_0}(I_{b_i}) \geq d_i \text{ for all } \tau \in T(A), \;\; i = 1, 2, ..., m, \tag{e 5.175}$$

where $\imath_0 : C(\mathbb{T}) \to A$ is defined by $\imath_0(f) = f(u)$ for all $f \in C(\mathbb{T})$, then one also has that

$$\mu_{\tau\circ\imath}(I_{c_i}) \geq (1-\epsilon_1)d_i \text{ for all } \tau \in T(A), \tag{e 5.176}$$

where I_{b_i} and I_{c_i} are any open arcs with length b_i and c_i, respectively, and where $c_i = b_i + \epsilon_1$, $i = 1, 2, ..., m$.

Proof. Let $\epsilon > 0$, and let $n \geq 64$ be an integer. Put $\epsilon_2 = \min\{\epsilon_1/16, 1/64n^2\}$. Let $\mathcal{F} \subset A$ be a finite subset and let $u \in U(A)$. Let $\delta_1 > 0$ (in place of δ) be as in 5.1 for ϵ, ϵ_2 (in place of ϵ_1) and let $\mathcal{G} = \{e_{i,j}\} \subset D \cong M_n$ be as required by 5.1.

Put $\delta = \delta_1/16$. By applying 5.4, there is a projection $p \in A$ and a C^*-subalgebra $D \cong M_n$ with $1_D = p$ such that

$$\|[x,p]\| \; < \; \delta \text{ for all } x \in \mathcal{F}; \tag{e 5.177}$$
$$\|[pxp, y]\| \; < \; \delta \text{ for all } x \in \mathcal{F} \text{ and } y \in D \text{ with } \|y\| \leq 1 \text{ and} \tag{e 5.178}$$
$$\tau(1-p) \; < \; \epsilon_2 \text{ for all } \tau \in T(C). \tag{e 5.179}$$

There is a unitary $u_0 \in (1-p)A(1-p)$ and a unitary $u_1 \in pAp$. Put $A_1 = pAp$ and $\mathcal{F}_1 = \{pxp : x \in \mathcal{F}\}$. We apply 5.1 to A_1, $\mathcal{F}_1$ and u_1. We check that lemma follows.

$\square$

The proof of the following lemma follows the same argument using 5.4 as in that of 5.5 but one applies 5.3 instead of 5.1.

Lemma 5.6. *Let $n \geq 64$ be an integer. Let $\epsilon > 0$ and $1/2 > \epsilon_1 > 0$. Suppose that A is a unital simple C^*-algebra with $TR(A) \leq 1$, X is a compact metric space, $\phi : C(X) \to A$ is a unital homomorphism, $\mathcal{F} \subset C(X)$ is a finite subset and suppose that $u \in U(A)$. Suppose also that, for some $\sigma > 0$ and $1 > b > 0$,*

$$\tau(\phi(f)) \geq \sigma \text{ for all } \tau \in T(A) \text{ and} \tag{e 5.180}$$

for all $f \in C(\mathbb{T})$ with $0 \le f \le 1$ whose supports contain an open ball with radius at least b. Then, there exists a unitary $v \in A$ and a continuous path of unitaries $\{v(t) : t \in [0,1]\} \subset A$ such that $v(0) = 1$, $v(1) = v$,

$$\|[\phi(f),\, v(t)]\| < \epsilon \ \text{ and } \ \|[u,\, v(t)]\| < \epsilon \ \text{ for all } f \in \mathcal{F} \text{ and } t \in [0,1], \tag{e 5.181}$$

$$\tau(\phi(f)g(vu)) \ge \frac{15\sigma}{24n^2} \ \text{ for all } \tau \in T(A) \tag{e 5.182}$$

for any $f \in C(X)$ with $0 \le f \le 1$ whose support contains an open ball of radius at least $2b$ and any $g \in C(\mathbb{T})$ with $0 \le g \le 1$ whose support contains an open arc of $\mathbb{T}$ with length $a \ge 8\pi/n$.

Moreover,

$$\mathrm{length}(\{v(t)\}) \le \pi. \tag{e 5.183}$$

If, in addition, $1 > b_1 > b_2 > \cdots > b_k > 0$, $1 > d_1 \ge d_2 \ge \cdots \ge d_k > 0$ are given and

$$\tau(\phi(f')g'(u)) \ge d_i \ \text{ for all } \tau \in T(A) \tag{e 5.184}$$

for any functions $f' \in C(X)$ with $0 \le f' \le 1$ whose support contains an open ball with radius $b_i/2$ and any function $g' \in C(\mathbb{T})$ with $0 \le g' \le 1$ whose support contains an arc with length b_i, then one also has that

$$\tau(\phi(f'')g''(vu)) \ge (1-\epsilon_1)d_i \ \text{ for all } \tau \in T(A), \tag{e 5.185}$$

where $f'' \in C(X)$ with $0 \le f'' \le 1$ whose support contains an open ball with radius c_i and $g'' \in C(\mathbb{T})$ with $0 \le g''$ whose support contains an arc with length $2c_i$, where $c_i = b_i + \epsilon_1$, $i = 1, 2, ..., k$.

5.7. Define

$$\Delta_{00}(r) = \frac{1}{2(n+1)^2} \ \text{ if } \ 0 < \frac{8\pi}{n+1} + \frac{4\pi}{2^{n+2}(n+1)} < r \le \frac{8\pi}{n} + \frac{4\pi}{2^{n+1}n} \tag{e 5.186}$$

$$\tag{e 5.187}$$

for $n \ge 64$ and

$$\Delta_{00}(r) = \frac{1}{2(65)^2} \ \text{ if } \ r \ge 8\pi/64 + \frac{4\pi}{2^{65}(64)}. \tag{e 5.188}$$

Let $\Delta : (0,1) \to (0,1)$ be a non-decreasing map. Define

$$D_0(\Delta)(r) = \Delta(\pi/n)\Delta_{00}(r) \ \text{ if } \ 0 < \frac{8\pi}{n+1} + \frac{4\pi}{2^{n+2}(n+1)} < r \le \frac{8\pi}{n} + \frac{4\pi}{2^{n+1}n} \tag{e 5.189}$$

$$\tag{e 5.190}$$

for $n \ge 64$ and

$$D_0(\Delta)(r) = D_0(\Delta)(4\pi/64) \ \text{ if } \ r \ge 8\pi/64 + \frac{4\pi}{2^{65}(64)}. \tag{e 5.191}$$

Lemma 5.8. *Suppose that A is a unital separable simple C^*-algebra with $TR(A) \le 1$, suppose that $\mathcal{F} \subset A$ is a finite subset and suppose that $u \in U(A)$. For any $\epsilon > 0$ and any $\eta > 0$, there exists a unitary $v \in U_0(A)$ and a continuous path of unitaries $\{w(t) : t \in [0,1]\} \subset U_0(A)$ such that*

$$w(0) = 1, \ w(1) = v, \qquad \|[f,\, w(t)]\| < \epsilon \ \text{ for all } f \in \mathcal{F} \text{ and } t \in [0,1], \ \text{ and } \tag{e 5.192}$$

$$\mu_{\tau\circ\imath}(I_a) \ge \Delta_{00}(a) \ \text{ for all } \tau \in T(A) \tag{e 5.193}$$

for any open arc I_a with length $a \ge \eta$, where $\imath : C(\mathbb{T}) \to A$ is defined by $\imath(g) = g(vu)$ for all $g \in C(\mathbb{T})$ and Δ_{00} is defined in 5.7.

Proof. Define

$$\Delta_{00,n}(r) = \frac{7}{12(k+1)^2} - \sum_{m=k}^{n} \frac{2}{9\cdot 2^{m+1}(m+1)^2} \tag{e 5.194}$$

$$\text{if } 0 < \frac{4\pi}{k+1} + \sum_{m=k}^{n} \frac{4\pi}{2^{m+1}2^{m+2}(m+1)} < r \le \frac{4\pi}{k} + \sum_{m=k}^{n} \frac{4\pi}{2^{m+1}2^{m+1}m} \tag{e 5.195}$$

if $n \ge k \ge 32$, and $\Delta_{00,n}(r) = \Delta_{00,n}(4\pi/32 + \frac{4\pi}{2^{32+1}32})$ if $r \ge 4\pi/32 + \frac{4\pi}{2^{32+1}32}$.

Without loss of generality, we may assume that $\eta = 4\pi/n$ for some $n \ge 32$. We will use the induction to prove the statement which is exactly the same as that of Lemma 5.8 but replace Δ_{00} by $\Delta_{00,k}$ for $k \ge 32$. It follows from 5.5, by choosing small ϵ_1, the statement holds for $k = 32$.

Now suppose that the statement holds for all integers m with $k \ge m \ge 32$. Thus we have a continuous path of unitaries $\{w'(t) : t \in [0,1]\} \subset U_0(A)$ such that

$$w'(0) = 1, \ \ w'(1) = v', \ \ \|[f,\, w'(t)]\| < \epsilon/2 \ \text{ for all } \ t \in [0,1] \ \text{ and} \tag{e 5.196}$$

$$\mu_{\tau\circ\imath_k}(I_a) \ge \Delta_{00,k} \ \text{ for all } \ \tau \in \mathrm{T}(A), \tag{e 5.197}$$

for all open arcs with length $a \ge 4\pi/k$, where $\imath_k : C(\mathbb{T}) \to A$ is defined by $\imath_k(g) = g(v'u)$ for all $g \in C(\mathbb{T})$.

Let

$$b_j = \frac{4\pi}{j+1} + \sum_{m=j}^{k} \frac{4\pi}{2^{m+1}2^{m+2}(m+1)} \ \text{ and } \ d_j = \frac{7}{12(j+1)^2} - \sum_{m=j}^{k} \frac{2}{9\cdot 2^{m+1}(m+1)^2} \tag{e 5.198}$$

$j = 32, 33, ..., k$. Choose $\epsilon_1 = \frac{2}{9\cdot 2^{k+2}(k+3)^2}$. By applying 5.5, we obtain a continuous path of unitaries $\{w''(t) : t \in [0,1]\} \subset U_0(A)$ such that

$$w''(0) = 1, \ \ w''(1) = v'', \ \ \|[f,\, w''(t)]\| < \epsilon/2 \ \text{ for all } \ t \in [0,1] \ \text{ and} \tag{e 5.199}$$

$$\mu_{\tau\circ\imath_{k+1}}(I_b) \ge \frac{15\pi}{24(k+1)^2} \ \text{ for all } \ \tau \in \mathrm{T}(A) \tag{e 5.200}$$

for all open arcs I_b with length $b \ge \frac{4\pi}{(k+1)}$, where $\imath_{k+1} : C(\mathbb{T}) \to A$ is defined by $\imath_{k+1}(g) = g(v''(v'u))$ for $g \in C(\mathbb{T})$. Moreover, for any open arc I_{c_j} with length c_j,

$$\tau\circ\imath_{\mathrm{k+1}}(I_{c_j}) \ge (1-\epsilon_1)d_j \ge \frac{7}{12(j+1)^2} - \sum_{m=j}^{k+1} \frac{2}{9\cdot 2^{m+1}(m+1)^2} \ \text{ for all } \ \tau \in \mathrm{T}(A), \tag{e 5.201}$$

$j = 32, 33, ..., k$. Now define $w(t) = w''(t)w'(t)$ for $t \in [0,1]$. Then

$$w(0) = 1, \ \ w(1) = v''v' \ \text{ and } \ \|[f,\, w(t)]\| < \epsilon \ \text{ for all } \ t \in [0,1]. \tag{e 5.202}$$

This shows that the statement holds for $k+1$. By the induction, this proves the statement.

Note that $\Delta_{00,n}(r) \ge \Delta_{00}(r)$ for all $r \ge 4\pi/n = \eta$. The lemma follows immediately from the statement.

∎

Corollary 5.9. *Let C be a unital separable simple amenable C^*-algebra with $TR(C) \le 1$ which satisfies the UCT. Let $\epsilon > 0$, $\mathcal{F} \subset C$ be a finite subset and let $1 > \eta > 0$.*

Suppose that A is a unital simple C^*-algebra with $TR(A) \le 1$, $\phi : C \to A$ is a unital homomorphism and $u \in U(A)$ is a unitary with

$$\|[\phi(c),\, u]\| < \epsilon \ \text{ for all } \ c \in \mathcal{F}. \tag{e 5.203}$$

Then there exist a continuous path of unitaries $\{u(t) : t \in [0,1]\} \subset U(A)$ such that

$$u(0) = u, \;\; u(1) = w \;\text{ and }\; \|[\phi(f),\, u(t)]\| < 2\epsilon \tag{e 5.204}$$

for all $f \in \mathcal{F}$ and $t \in [0,1]$. Moreover, for any open arc I_a with length a,

$$\mu_{\tau\circ\imath}(I_a) \ge \Delta_{00}(a) \;\textit{ for all }\; a \ge \eta, \tag{e 5.205}$$

where $\imath : C(\mathbb{T}) \to A$ is defined by $\imath(f) = f(w)$ for all $f \in C(\mathbb{T})$.

Proof. Let $\epsilon > 0$ and $\mathcal{F} \subset C$ be as described. Put $\mathcal{F}_1 = \phi(\mathcal{F})$. The corollary follows from 5.8 by taking $u(t) = w(t)u$.

□

The proof of the following lemma follows from the same argument used in that of 5.8 by applying 5.6 instead.

Lemma 5.10. *Let $\Delta : (0,1) \to (0,1)$ be a non-decreasing map, let $\eta > 0$, let X be a compact metric space and let $\mathcal{F} \subset C(X)$ be a finite subset. Suppose that A is a unital simple C^*-algebra with $TR(A) \le 1$, suppose that $\phi : C(X) \to A$ is a unital homomorphism and suppose that $u \in U(A)$ such that*

$$\mu_{\tau\circ\phi}(O_a) \ge \Delta(a) \;\textit{ for all }\; \tau \in T(A) \tag{e 5.206}$$

for any open ball with radius $a \ge \eta$. For any $\epsilon > 0$, there exists a unitary $v \in U_0(A)$ and a continuous path of unitaries $\{v(t) : t \in [0,1]\} \subset U_0(A)$ such that

$$v(0) = 1, \;\; v(1) = v \tag{e 5.207}$$
$$\|[\phi(f),\, v(t)]\| < \epsilon, \;\; \|[u,\, v(t)]\| < \epsilon \;\textit{ for all }\; f \in \mathcal{F} \;\textit{ and }\; t \in [0,1] \;\textit{ and} \tag{e 5.208}$$
$$\tau(\phi(f)g(vu)) \ge D_0(\Delta)(a) \;\textit{ for all }\; \tau \in T(A) \tag{e 5.209}$$

for any $f \in C(X)$ with $0 \le f \le 1$ whose support contains an open ball with radius $a \ge 4\eta$ and any $g \in C(\mathbb{T})$ with $0 \le g \le 1$ whose support contains an open arc with length $a \ge 4\eta$, where $D_0(\Delta)$ is defined in 5.7.

6 The Basic Homotopy Lemma for $C(X)$

In this section we will prove Theorem 6.2 below. We will apply the results of the previous section to produce the map L which was required in Theorem 4.5 by using a continuous path of unitaries.

Lemma 6.1. *Let X be a compact metric space, let $\Delta : (0,1) \to (0,1)$ be a non-decreasing map, let $\epsilon > 0$, let $\eta > 0$ and let $\mathcal{F} \subset C(X)$ be a finite subset. There exists $\delta > 0$ and a finite subset $\mathcal{G} \subset C(X)$ satisfying the following:*

Suppose that A is a unital simple C^*-algebra with $TR(A) \le 1$, suppose that $\phi : C(X) \to A$ and suppose that $u \in U(A)$ such that

$$\|[\phi(f),\, u]\| < \delta \;\textit{ for all }\; f \in \mathcal{G} \;\textit{ and} \tag{e 6.210}$$

$$\mu_{\tau\circ\phi}(O_b) \ge \Delta(a) \;\textit{ for all }\; \tau \in T(A) \tag{e 6.211}$$

for any open balls O_b with radius $b \ge \eta/2$. There exists a unitary $v \in U_0(A)$, a unital completely positive linear map $L : C(X \times \mathbb{T}) \to A$ and a continuous path of unitaries $\{v(t) : t \in [0,1]\} \subset U_0(A)$ such that

$$v(0) = u, \;\; v(1) = v, \;\; \|[\phi(f),\, v(t)]\| < \epsilon \text{ for all } f \in \mathcal{F} \text{ and } t \in [0,1], \qquad \text{(e 6.212)}$$
$$\|L(f \otimes z) - \phi(f)v\| < \epsilon, \;\; \|L(f \otimes 1) - \phi(f)\| < \epsilon \text{ for all } f \in \mathcal{F} \text{ and} \qquad \text{(e 6.213)}$$
$$\mu_{\tau \circ L}(O_a) \ge (2/3)D_0(\Delta)(a/2) \text{ for all } \tau \in T(A) \qquad \text{(e 6.214)}$$

for any open balls O_a of $X \times \mathbb{T}$ with radius $a \ge 5\eta$, where $D_0(\Delta)$ is defined in 5.7.

Proof. Fix $\epsilon > 0$, $\eta > 0$ and a finite subset $\mathcal{F} \subset C(X)$. Let $\mathcal{F}_1 \subset C(X)$ be a finite subset containing $\mathcal{F}$. Let $\epsilon_0 = \min\{\epsilon/2, D_0(\Delta)(\eta)/4\}$. Let $\mathcal{G} \subset C(X)$ be a finite subset containing $\mathcal{F}$, $1_{C(X)}$ and z. There is $\delta_0 > 0$ such that there is a unital completely positive linear map $L' : C(X \times \mathbb{T}) \to B$ (for unital C^*-algebra B) satisfying the following:

$$\|L'(f \otimes z) - \phi'(f)u'\| < \epsilon_0 \text{ for all } f \in \mathcal{F}_1 \qquad \text{(e 6.215)}$$

for any unital homomorphism $\phi' : C(X) \to B$ and any unitary $u' \in B$ whenever

$$\|[\phi'(g),\, u']\| < \delta_0 \text{ for all } g \in \mathcal{G}. \qquad \text{(e 6.216)}$$

Let $0 < \delta < \min\{\delta_0/2, \epsilon/2, \epsilon_0/2\}$ and suppose that

$$\|[\phi(g),\, u]\| < \delta \text{ for all } g \in \mathcal{G}. \qquad \text{(e 6.217)}$$

It follows from 5.10 that there is a continuous path of unitaries $\{z(t) : t \in [0,1]\} \subset U_0(A)$ such that

$$z(0) = 1, \;\; z(1) = v_1, \qquad \text{(e 6.218)}$$
$$\|[\phi(f),\, z(t)]\| < \delta/2, \;\; \|[u,\, z(t)]\| < \delta/2 \text{ for all } t \in [0,1] \text{ and} \qquad \text{(e 6.219)}$$
$$\tau(\phi(f)g(v_1 u)) \ge D_0(\Delta)(a) \qquad \text{(e 6.220)}$$

for any $f \in C(X)$ with $0 \le f \le 1$ whose support contains an open ball with radius 4η and $g \in C(\mathbb{T})$ with $0 \le g \le 1$ whose support contains open arcs with length $a \ge 4\eta$.

Put $v = v_1 u$. Then we obtain a unital completely positive linear map $L : C(X \times \mathbb{T}) \to A$ such that

$$\|L(f \otimes z) - \phi(f)v\| < \epsilon_0 \text{ and } \|L(f \otimes 1) - \phi(f)\| < \epsilon_0 \text{ for all } f \in \mathcal{F}_1. \qquad \text{(e 6.221)}$$

If $\mathcal{F}_1$ is sufficiently large (depending on η only), we may also assume that

$$\mu_{\tau \circ L}(B_a \times J_a) \ge (2/3)D_0(\Delta)(a/2) \qquad \text{(e 6.222)}$$

for any open ball B_a with radius a and open arcs with length a, where $a \ge 5\eta$.

∎

Theorem 6.2. *Let X be a finite CW complex so that $X \times \mathbb{T}$ has the property* (H)*. Let $C = PC(X, M_n)P$ for some projection $P \in C(X, M_n)$ and let $\Delta : (0,1) \to (0,1)$ be a non-decreasing map. For any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset C$, there exists $\delta > 0$, $\eta > 0$ and there exists a finite subset $\mathcal{G} \subset C$ satisfying the following:*

Suppose that A is a unital simple C^*-algebra with $TR(A) \le 1$, $\phi : C \to A$ is a unital homomorphism and $u \in A$ is a unitary and suppose that

$$\|[\phi(c),\, u]\| < \delta \text{ for all } c \in \mathcal{G} \text{ and } \mathrm{Bott}(\phi, u) = \{0\}. \qquad \text{(e 6.223)}$$

Suppose also that

$$\mu_{\tau\circ\phi}(O_a) \geq \Delta(a) \tag{e 6.224}$$

for all open balls O_a of X with radius $1 > a \geq \eta$, where $\mu_{\tau\circ\phi}$ is the Borel probability measure defined by restricting ϕ on the center of C. Then there exists a continuous path of unitaries $\{u(t) : t \in [0,1]\}$ in A such that

$$u(0) = u, \;\; u(1) = 1 \;\; \textit{and} \;\; \|[\phi(c),\, u(t)]\| < \epsilon \tag{e 6.225}$$

for all $c \in \mathcal{F}$ and for all $t \in [0,1]$.

Proof. First it is easy to see that the general case can be reduced to the case that $C = C(X, M_n)$. It is then easy to see that this case can be further reduced to the case that $C = C(X)$. Then the theorem follows from the combination of 4.4 and 6.1.

□

Corollary 6.3. *Let $k \geq 1$ be an integer, let $\epsilon > 0$ and let $\Delta : (0,1) \to (0,1)$ be any non-decreasing map. There exist $\delta > 0$ and $\eta > 0$ (η does not depend on Δ) satisfying the following: For any k mutually commutative unitaries $u_1, u_2, ..., u_k$ and a unitary $v \in U(A)$ in a unital separable simple C^*-algebra A with tracial rank no more than one for which*

$$\|[u_i,\, v]\| < \delta, \; \mathrm{bott}_j(u_i, v) = 0, \;\; j = 0, 1, \; i = 1, 2, ..., k, \;\; \textit{and} \;\; \mu_{\tau\circ\phi}(O_a) \geq \Delta(a) \;\; \textit{for all} \;\; \tau \in T(A),$$

for any open ball O_a with radius $a \geq \eta$, where $\phi : C(\mathbb{T}^k) \to A$ is the homomorphism defined by $\phi(f) = f(u_1, u_2, ..., u_k)$ for all $f \in C(\mathbb{T}^k)$, there exists a continuous path of unitaries $\{v(t) : t \in [0,1]\} \subset A$ such that $v(0) = v$, $v(1) = 1$ and

$$\|[u_i,\, v(t)]\| < \epsilon \;\; \textit{for all} \;\; t \in [0,1], \quad i = 1, 2, ..., k.$$

Remark 6.4. In 6.3, if $k = 1$, the condition that $\mathrm{bott}_0(u_1, v) = 0$ is the same as $v \in U_0(A)$. Note that in Theorem 6.2, the constant δ depends not only on ϵ and the finite subset $\mathcal{F}$ but also depends on the measure distribution Δ. As in section 9 of [29], in general, δ can not be chosen independent of Δ.

Unlike the Basic Homotopy Lemma in simple C^*-algebras of real rank zero, in Theorem 6.2 as well as in 6.3, the length of $\{u(t)\}$ (or $\{v_t\}$) can not be possibly controlled. To see this, one notes that, it is known (see [39]) that $\mathrm{cel}(A) = \infty$ for some simple AH-algebras with no dimension growth. It is proved (see [13], or Theorem 2.5 of [28]) that all of these C^*-algebras A have tracial rank one. For those simple C^*-algebras, let $k = 1$. For any number $L > \pi$, choose $u = v$ and $v \in U_0(A)$ with $\mathrm{cel}(v) > L$. This gives an example that the length of $\{v_t\}$ is longer than L. This shows that, in general, the length of $\{v_t\}$ could be as long as one wishes.

However, we can always assume that the path $\{u(t) : t \in [0,1]\}$ is piece-wise smooth. For example, suppose that $\{u(t) : t \in [0,1]\}$ satisfies the conclusion of 6.2 for $\epsilon/2$. There are $0 = t_0 < t_1 < \cdots t_n = 1$ such that

$$\|u(t_i) - u(t_{i-1})\| < \epsilon/32, \;\; i = 1, 2, ..., n.$$

There is a selfadjoint element $h_i \in A$ with $\|h_i\| \leq \epsilon/8$ such that

$$u(t_i) = u(t_{i-1}) \exp(\sqrt{-1}h_i), \;\; i = 1, 2, ..., n.$$

Define

$$w(t) = u(t_{i-1}) \exp(\sqrt{-1}(\frac{t - t_{i-1}}{t_i - t_{i-1}})h_i) \;\; \text{for all} \;\; t \in [t_{i-1}, t_i),$$

$i = 1, 2, ..., n$. Note that

$$\|[\phi(c),\, w(t)]\| < \epsilon \;\; \text{for all} \;\; t \in [0,1].$$

On the other hand, it is easy to see that $w(t)$ is continuous and piece-wise smooth.

7 An approximate unitary equivalence result

The following is a variation of some results in [15]. We refer to [15] for the terminologies used in the following statement.

Theorem 7.1. (cf. Theorem 1.1 of [15]) *Let C be a unital separable amenable C^*-algebra satisfying the UCT. Let $b \ge 1$, let $T : \mathbb{N}^2 \to \mathbb{N}$, $L : U(M_\infty(C)) \to \mathbb{R}_+$, $E : \mathbb{R}_+ \times \mathbb{N} \to \mathbb{R}_+$ and $T_1 = N \times K : C_+ \setminus \{0\} \to \mathbb{N} \times \mathbb{R}_+ \setminus \{0\}$ be four maps. For any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset C$, there exists $\delta > 0$, a finite subset $\mathcal{G} \subset C$, a finite subset $\mathcal{H} \subset C_+ \setminus \{0\}$, a finite subset $\mathcal{P} \subset \underline{K}(C)$, a finite subset $\mathcal{U} \subset U(M_\infty(C))$, an integer $l > 0$ and an integer $k > 0$ satisfying the following:*

for any unital C^*-algebra A with stable rank one, K_0-divisible rank T, exponential length divisible rank E and $\mathrm{cer}(\mathrm{M_m(A)}) \le \mathrm{b}$ (for all m), if $\phi, \psi : C \to A$ are two unital δ-$\mathcal{G}$-multiplicative contractive completely positive linear maps with

$$[\phi]|_{\mathcal{P}} = [\psi]|_{\mathcal{P}} \text{ and } \mathrm{cel}(\langle\phi\rangle(u)^*\langle\psi\rangle(u)) \le L(u) \tag{e 7.226}$$

for all $u \in \mathcal{U}$, then for any unital δ-$\mathcal{G}$-multiplicative contractive completely positive linear map $\theta : C \to M_l(A)$ which is also T-$\mathcal{H}$-full, there exists a unitary $u \in M_{lk+1}(A)$ such that

$$\|u^*\mathrm{diag}(\phi(a), \overbrace{\theta(a), \theta(a), \cdots, \theta(a)}^{k})u - \mathrm{diag}(\psi(a), \overbrace{\theta(a), \theta(a), \cdots, \theta(a)}^{k})\| < \epsilon \tag{e 7.227}$$

for all $a \in \mathcal{F}$.

Proof. Suppose that the theorem is false. Then there exists $\epsilon_0 > 0$ and a finite subset $\mathcal{F} \subset C$ such that there are a sequence of positive numbers $\{\delta_n\}$ with $\delta_n \downarrow 0$, an increasing sequence of finite subsets $\{\mathcal{G}_n\}$ whose union is dense in C, an increasing sequence of finite subsets $\{\mathcal{H}_n\} \subset C_+ \setminus \{0\}$ whose union is dense in C_+, a sequence of finite subsets $\{\mathcal{P}_n\}$ of $\underline{K}(C)$ with $\cup_{n=1}^\infty \mathcal{P}_n = \underline{K}(C)$, a sequence of finite subsets $\{\mathcal{U}_n\} \subset U(M_\infty(C))$, two sequences of $\{l(n)\}$ and $\{k(n)\}$ of integers (with $\lim_{n\to\infty} l(n) = \infty$), a sequence of unital C^*-algebra A_n with stable rank one, K_0-divisible rank T, exponential length divisible rank E and $\mathrm{cer}(M_m(A_n)) \le b$ (for all m) and sequences $\{\phi_n\}$, $\{\psi_n\}$ of $\mathcal{G}_n$-δ_n-multiplicative contractive completely positive linear maps from C into A_n with

$$[\phi_n]|_{\mathcal{P}} = [\psi_n]|_{\mathcal{P}} \text{ and } \mathrm{cel}(\langle\phi\rangle(u)\langle\psi\rangle(u^*)) \le L(u) \tag{e 7.228}$$

$u \in \mathcal{U}_n$ satisfying the following:

$$\inf\{\sup\{\|v^*\mathrm{diag}(\phi_n(a), S_n(a))v - \mathrm{diag}(\psi_n(a), S_n(a))\| : a \in \mathcal{F}\} \ge \epsilon_0 \tag{e 7.229}$$

where the infimum is taken among all unital T_1-$\mathcal{H}_n$-full and δ_n-$\mathcal{G}_n$-multiplicative contractive completely positive linear maps $\sigma_n : C \to M_{l(n)}(A_n)$ and where

$$S_n(a) = \mathrm{diag}(\overbrace{\sigma_n(a), \sigma_n(a), \cdots, \sigma_n(a)}^{k(n)}),$$

and among all unitaries v in $M_{l(n)k(n)+1}(A_n)$.

Let $B_0 = \bigoplus_{n=1}^\infty A_n$, $B = \prod_{n=1}^\infty B_n$, $Q(B) = B/B_0$ and $\pi : B \to Q(B)$ be the quotient map. Define $\Phi, \Psi : C \to B$ by $\Phi(a) = \{\phi_n(a)\}$ and $\Psi(a) = \{\psi_n(a)\}$ for $a \in C$. Note that $\pi \circ \Phi$ and $\pi \circ \Psi$ are homomorphism.

For any $u \in \mathcal{U}_m$, since A_n has stable rank one, when $n \ge m$,

$$\langle\phi_n\rangle(u)(\langle\psi_n\rangle(u))^* \in U_0(A_n) \text{ and } \mathrm{cel}(\langle\phi_n\rangle(u)(\langle\psi_n\rangle(u))^*) \le L(u). \tag{e 7.230}$$

It follows that, for all $n \geq m$, (by Lemma 1.1 of [15] for example), there is a continuous path $\{U(t) \in \prod_{n=m}^{\infty} A_n : t \in [0,1]\}$ such that

$$U(0) = \{\langle \phi_n \rangle(u)\}_{n \geq m} \text{ and } U(1) = \{\langle \psi_n \rangle(u)\}_{n \geq m}.$$

Since this holds for each m, it follows that

$$(\pi \circ \Phi)_{*1} = (\pi \circ \Psi)_{*1} \tag{e 7.231}$$

It follows from (2) of Corollary 2.1 of [15] that

$$K_0(B) = \prod_b K_0(B_n) \text{ and } K_0(Q(B)) = \prod_b K_0(B_n) / \bigoplus_n K_0(B_n). \tag{e 7.232}$$

Then, by (e 7.228) and by using the fact that each B_n has stable rank one again, one concludes that

$$(\pi \circ \Phi)_{*0} = (\pi \circ \Psi)_{*0} \tag{e 7.233}$$

Moreover, with the same argument, by (e 7.228) and by applying (2) of Corollary 2.1 of [15],

$$[\pi \circ \Phi]|_{K_i(C,\mathbb{Z}/k\mathbb{Z})} = [\pi \circ \Psi]|_{K_i(C,\mathbb{Z}/k\mathbb{Z})}, \;\; k = 2, 3, ..., \text{ and } i = 0, 1. \tag{e 7.234}$$

Since C satisfies the UCT, by [7],

$$[\pi \circ \Phi] = [\pi \circ \Psi] \text{ in } KL(C, Q(B)). \tag{e 7.235}$$

On the other hand, since each σ_n is δ_n-$\mathcal{G}_n$-multiplicative and T_1-$\mathcal{H}_n$-full, we conclude that $\pi \circ \Sigma$ is a full homomorphism, where $\Sigma : C \to B$ is defined by $\Sigma(c) = \{\sigma_n(c)\}$ for $c \in C$.

It follows from Theorem 3.9 of [25] that there exists an integer N and a unitary $\bar{W} \in Q(B)$ such that

$$\|\bar{W}^* \mathrm{diag}(\pi \circ \Phi(c), \overbrace{\pi \circ \Sigma(c), \cdots, \pi \circ \Sigma(c)}^{N})\bar{W} \tag{e 7.236}$$

$$-\mathrm{diag}(\pi \circ \Psi(c), \pi \circ \overbrace{\pi \circ \Sigma(c), \cdots, \pi \circ \Sigma(c)}^{N})\| < \epsilon_0/2 \tag{e 7.237}$$

for all $c \in \mathcal{F}$. There exists a unitary $u_n \in A_n$ for each n such that $\pi(\{u_n\}) = \bar{W}$. Therefore, by (e 7.238), for some large $n_0 \geq 0$,

$$\|u_n^* \mathrm{diag}(\phi_n(c), \overbrace{\sigma_n(c), \cdots, \sigma_n(c)}^{N})u_n \tag{e 7.238}$$

$$-\mathrm{diag}(\psi_n(c), \overbrace{\sigma_n(c), \cdots, \sigma_n(c)}^{N})\| < \epsilon_0 \tag{e 7.239}$$

for all $c \in \mathcal{F}$. This contradicts with (e 7.229).

□

Remark 7.2. Suppose that $U(C)/U_0(C) = K_1(C)$. Then, from the proof, one sees that we may only consider $\mathcal{U} \subset U(C)$.

Theorem 7.3. *Let C be a unital separable simple amenable C^*-algebra with $TR(C) \le 1$ satisfying the UCT and let $D = C \otimes C(\mathbb{T})$. Let $T = N \times K : D_+ \setminus \{0\} \to \mathbb{N}_+ \times \mathbb{R}_+ \setminus \{0\}$.*

Then, for any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset D$, there exists $\delta > 0$, a finite subset $\mathcal{G} \subset D$, a finite subset $\mathcal{H} \subset D_+ \setminus \{0\}$, a finite subset $\mathcal{P} \subset \underline{K}(C)$ and a finite subset $\mathcal{U} \subset U(D)$ satisfying the following: Suppose that A is a unital simple C^*-algebra with $TR(A) \le 1$ and $\phi, \psi : D \to A$ are two unital δ-$\mathcal{G}$-multiplicative contractive completely positive linear maps such that ϕ, ψ are T-$\mathcal{H}$-full,

$$|\tau \circ \phi(g) - \tau \circ \psi(g)| < \delta \ \text{for all} \ g \in \mathcal{G} \tag{e 7.240}$$

for all $\tau \in T(A)$,

$$[\phi]|_{\mathcal{P}} = [\psi]|_{\mathcal{P}} \ \text{and} \tag{e 7.241}$$
$$\mathrm{dist}(\phi^{\ddagger}(\bar{w}), \psi^{\ddagger}(\bar{w})) < \delta \tag{e 7.242}$$

for all $w \in \mathcal{U}$. Then there exists a unitary $u \in U(A)$ such that

$$\mathrm{ad}\, u \circ \psi \approx_{\epsilon} \phi \ \text{on} \ \mathcal{F}. \tag{e 7.243}$$

Proof. Let $\epsilon > 0$ and a finite subset $\mathcal{F} \subset D$. Fix $T = N \times K$ as given. Let $T_1 = N \times 2K$. Let $1 > \delta_1 > 0$, $\mathcal{G}_1 \subset D$, $\mathcal{H}_1 \subset D_+ \setminus \{0\}$, $\mathcal{P}_1 \subset \underline{K}(D)$, $\mathcal{U}_1 \subset U(M_\infty(D))$, integer l and k as required by 7.1 for $\epsilon/8$, $\mathcal{F}$ and T as well as for $b = 2$, $T(n,m) = 1$, $L(u) = 2\mathrm{cel}(u) + 8\pi + 1$, $E(l,k) = 8\pi + l/k$. We may assume that $\delta_1 < \min\{\epsilon/8, 1/8\pi\}$ and $k \ge 2$. Without loss of generality, we may assume that $\mathcal{G}_1 = \{g \otimes 1, g \in \mathcal{G}_0\} \cup \{1 \otimes z\}$, where $\mathcal{G}_0 \in C$ and z is the identity function on $\mathbb{T}$, the unit circle. Note that $K_1(D) = K_1(A) \bigoplus K_0(A)$. It is clear that $K_1(D)$ is generated by $u \otimes 1$ and $(p \otimes z) + (1-p) \otimes 1$ for $u \in U(A)$ and projections $p \in A$. In particular, $K_1(D) = U(D)/U_0(D)$. Thus (see the remark 7.2), we may assume that $\mathcal{U}_1 \subset U(A)$.

Since $TR(C) \le 1$, for any $\delta_2 > 0$, there exists a projection $e \in C$ and a C^*-subalgebra $C_0 \in \mathcal{I}$ with $1_{C_0} = e$ and a contractive completely positive linear map $j_1 : C \to C_0$ such that

(1) $\|[x, e]\| < \delta_2$ for $x \in \mathcal{G}_0$;

(2) $\mathrm{dist}(exe, j_1(x)) < \delta_2/4$ for $x \in \mathcal{G}_0$ and

(3) $(2kl+1)\tau(1-e) < \tau(e)$ and $\tau(1-e) < \delta_2/(2kl+1)$ for all $\tau \in T(C)$.

Put $z_0 = (1-e) \otimes z$, $z_1 = e \otimes z$ and $j_0(c) = (1-e)c(1-e)$ for $c \in C$. We may also assume that $\delta_2 < \delta_1/4$. Put $\mathcal{G}_{00} = j_1(\mathcal{G}_0)$. Thus

$$\mathrm{dist}(exe, \mathcal{G}_{00}) < \delta/4 \ \text{for all} \ x \in \mathcal{G}_0. \tag{e 7.244}$$

Let $D_0 = C_0 \otimes C(\mathbb{T})$. Let $T' = T|_{(D_0)_+ \setminus \{0\}}$. Let $\delta_3 > 0$ (in place δ), let $\mathcal{G}_2$ (in place of $\mathcal{G}$) be a finite subset of D_0, let $\mathcal{H}_2 \subset (D_0)_+ \setminus \{0\}$, let $\mathcal{P}_2$ (in place of $\mathcal{P}$) be a finite subset of $\underline{K}(D_0)$ and let $\mathcal{U}_2$ (in place of $\mathcal{U}$) be a finite subset of $U(M_\infty(D_0))$ required by Theorem 11.5 of [34] for $\delta_1/4$ (in place of ϵ), $\mathcal{G}_{00} \cup \{z_1\}$ (in place of $\mathcal{F}$) (and D_0 in place of C). Here we identify e with 1_{D_0}. Let $J = j_1 \otimes \mathrm{id}_{C(\mathbb{T})} : D_0 \to D$ be the obvious embedding and $J_0 = j_0 \otimes \mathrm{id}_{C(\mathbb{T})}$. Let $\mathcal{P}_2' \in \underline{K}(D)$ be the image of $\mathcal{P}_2$ under $[J]$.

Now let $\delta = \min\{\delta_2/(8kl+1), \delta_3/(8kl+1)\}$, $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2 \cup \{e, (1-e)\}$. Here we also view $\mathcal{G}_2$ as a subset of D. Let $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2$, let $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2'$ and $\mathcal{U} = \mathcal{U}_1 \cup \{(e + \langle j_0 \rangle(u) : u \in \mathcal{U}_1\} \cup \{(1-e) + v : v \in \mathcal{U}_2\}$.

Suppose that ϕ and ψ satisfy the assumptions of the theorem for the above $\mathcal{G}$, $\mathcal{H}$, $\mathcal{P}$ and $\mathcal{U}$. Let $\phi' = \phi \circ J$, $\psi' = \psi \circ J$. There is a unitary $u_0 \in A$ such that

$$u_0^* \psi'(e) u_0 = \phi'(e) = e_0 \in A.$$

Put $A_1 = e_0Ae_0$. We have $[\mathrm{ad}\, u_0 \circ \psi']|_{\mathcal{P}_2} = [\phi']|_{\mathcal{P}_2}$ and, for $g \in \mathcal{G}$,

$$|t \circ \mathrm{ad}\, u_0 \circ \psi'(g) - t \circ \phi'(g)| \quad < \quad \frac{\delta}{1 - \delta/(2kl+1)} < \delta_3 \text{ for all } t \in T(eAe). \quad \text{(e 7.245)}$$

Moreover, by the first part of 3.3,

$$\mathrm{dist}((\mathrm{ad}\, u_0 \circ \psi')^{\ddagger}(\bar{w}), (\phi')^{\ddagger}(\bar{w})) < (2+1)\delta < \delta_3 \quad \text{(e 7.246)}$$

for all $w \in \mathcal{U}_2$.

By the choice of $\mathcal{G}_2$, $\mathcal{H}_2, \mathcal{U}_2$ and $\mathcal{P}_2$, and by applying 11.5 of [34], there is a unitary $u_1 \in A_1$ such that

$$\mathrm{ad}\, u_1 \circ \mathrm{ad}\, u_0 \circ \psi' \approx_{\epsilon/2} \phi' \text{ on } \mathcal{G}_{00}. \quad \text{(e 7.247)}$$

Let $\mathcal{G}_{00}'$ be a finite subset containing $\mathcal{G}_{00} \cup j(\mathcal{H}_1)$ and $\delta_4 > 0$. Since $TR(A_1) \le 1$, by Lemma 5.5 of [28], there are mutually orthogonal projections $q_0, q_1, q_2, ..., q_{8kl+4}$ with $[q_0] \le [q_1]$ and $[q_1] = [q_i]$, $i = 1, 2, ..., 8kl+4$, and there are unital δ_4-$\mathcal{G}_{00}'$ multiplicative contractive completely positive linear maps $L_0 : D_0 \to q_0Aq_0$ and $L_i : D_0 \to q_iAq_i$ ($i = 1, 2, ..., 8kl+4$) such that

$$\phi' \approx_{\delta_4} L_0 \oplus L_1 \oplus L_2 \oplus \cdots \oplus L_{8kl+4} \text{ on } \mathcal{G}_{00}'', \quad \text{(e 7.248)}$$

and there exists a unitary $W_i \in (q_1 + q_i)A(q_1 + q_i)$ such that

$$\mathrm{ad}W_i \circ L_i = L_1, \;\; i = 1, 2, ..., 8kl. \quad \text{(e 7.249)}$$

Since ϕ is T-$\mathcal{H}$-full, with sufficiently small δ_4 and sufficiently large $\mathcal{G}_{00}'$ we may also assume that each $L_i \circ j_1$ is also T_1-$\mathcal{H}_1$-full and $\delta_4 < \delta/4$. Define $Q_i = \sum_{j=4+8(i-1)}^{8i+4} q_i$, $Q_0 = \sum_{i=1}^{4} q_i$ and $\Phi_i = \sum_{j=4+8(i-1)}^{8i+4} L_i$, $i = 1, 2, ..., kl$. Note by e 7.249), Φ_i are unitarily equivalent to Φ_1.

Since $K_0(A)$ is weakly unperforated (see Theorem 6.11 of [19]), we check that

$$[p_0 + q_0 + Q_0] \le [Q_i] \text{ and } 2[p_0 + q_0 + Q_0] \ge [Q_i], \;\; i = 1, 2, ..., kl. \quad \text{(e 7.250)}$$

Put $\phi_0 = \phi \circ J_0 \oplus L_0 \circ J \oplus \sum_{i=1}^{4} L_i \circ J$ and $\psi_0 = \psi \circ J_0 \oplus L_0 \circ J \oplus \sum_{i=1}^{4} L_i \circ J$. By 3.3, we compute that

$$\mathrm{dist}(\phi_0^{\ddagger}(\bar{w}), \psi_0^{\ddagger}(\bar{w})) < \delta_1 \text{ for all } w \in \mathcal{U} \quad \text{(e 7.251)}$$

It follows Lemma 6.9 of [28] that

$$\mathrm{cel}(\langle \phi_0 \rangle(u) \langle \psi_0 \rangle(u)^*) < 8\pi + 1 \text{ for all } u \in \mathcal{U}. \quad \text{(e 7.252)}$$

We also have

$$[\phi_0]|_{\mathcal{P}_1} = [\psi_0]|_{\mathcal{P}_1}. \quad \text{(e 7.253)}$$

Since $\Phi_1 \circ j$ is T_1-$\mathcal{H}_1$-full, by applying 7.1, we obtain a unitary $w \in U(A)$,

$$\|w^*\mathrm{diag}(\psi_0(c), \overbrace{\Phi_1 \circ J(c), ..., \Phi_1 \circ J(c)}^{kl})w - \mathrm{diag}(\phi_0(c), \overbrace{\Phi_1 \circ J(c), ..., \Phi_1 \circ J(c)}^{kl})\| < \epsilon/8 \quad \text{(e 7.254)}$$

for all $c \in \mathcal{F}$. Since $\Phi_i \circ j_1$ is unitarily equivalent to $\Phi_1 \circ j_1$, there is a unitary $w' \in U(A)$ such that

$$\|(w')^*\mathrm{diag}(\psi_0(c), \Phi_1 \circ J(c), ..., \Phi_{kl} \circ J(c))w' \quad \text{(e 7.255)}$$

$$-\mathrm{diag}(\phi_0(c), \Phi_1 \circ J(c), ..., \Phi_{kl} \circ J(c))\| < \epsilon/8 \quad \text{(e 7.256)}$$

for all $c \in \mathcal{F}$. It follows that

$$\|(w')^* \mathrm{diag}(\psi \circ J_0(c), L_0 \circ J(c), \phi'(c))w' - \mathrm{diag}(\phi \circ J_0(c), L_0 \circ J(c), \phi'(c))\| < \epsilon/4 \quad (e\,7.257)$$

for all $c \in \mathcal{F}$.

Let $u = ((1 - e_0) \oplus e_0 u_0 u_1)w'$. Then, by (e 7.247), we have

$$\|u^* \mathrm{diag}(\psi \circ J_0(c), \psi'(c))u - \mathrm{diag}(\phi \circ J_0(c), \phi'(c))\| < \epsilon/2 \quad (e\,7.258)$$

for all $c \in \mathcal{F}$. It follows that

$$\mathrm{ad}\, u \circ \psi \approx_\epsilon \phi \ \text{ on } \ \mathcal{F}. \quad (e\,7.259)$$

□

Corollary 7.4. *Let C be a unital separable amenable simple C^*-algebra with $TR(C) \le 1$ which satisfies the UCT, let $D = C \otimes C(\mathbb{T})$ and let A be a unital simple C^*-algebra with $TR(A) \le 1$. Suppose that $\phi, \psi : D \to A$ are two unital monomorphisms. Then ϕ and ψ are approximately unitarily equivalent, i.e., there exists a sequence of unitaries $\{u_n\} \subset A$ such that*

$$\lim_{n\to\infty} \mathrm{ad}\, u_n \circ \psi(d) = \phi(d) \ \textit{ for all } \ d \in D,$$

if and only if

$$[\phi] = [\psi] \ \text{ in } \ KL(D, A), \tau \circ \phi = \tau \circ \psi \ \textit{ for all } \ \tau \in T(A) \ \text{ and } \ \psi^\ddagger = \phi^\ddagger.$$

8 The Main Basic Homotopy Lemma

Lemma 8.1. *Let C be a unital separable simple C^*-algebra with $TR(C) \le 1$ and let $\Delta : (0, 1) \to (0, 1)$ be a non-decreasing map. There exists a map $T = N \times K : D_+ \setminus \{0\} \to \mathbb{N}_+ \times \mathbb{R}_+ \setminus \{0\}$, where $D = C \otimes C(\mathbb{T})$, satisfying the following:*

For any $\epsilon > 0$, any finite subset $\mathcal{F} \subset C$ and any finite subset $\mathcal{H} \subset D_+ \setminus \{0\}$, there exists $\delta > 0$, $\eta > 0$ and a finite subset $\mathcal{G} \subset C$ satisfying the following: for any unital separable unital simple C^*-algebra A, any unital homomorphism $\phi : C \to A$ and any unitary $u \in A$ such that

$$\|[\phi(c), u]\| < \delta \ \textit{ for all } \ c \in \mathcal{G} \ \textit{ and} \quad (e\,8.260)$$

$$\mu_{\tau\circ\imath}(O_a) \ge \Delta(a) \ \textit{ for all } \ \tau \in T(A) \quad (e\,8.261)$$

and for all open balls O_a with radius $a \ge \eta$, where $\imath : C(\mathbb{T}) \to A$ is defined by $\imath(f) = f(u)$, there is a unital contractive completely positive linear map $L : D \to A$ such that

$$\|L(c \otimes 1) - \phi(c)\| < \epsilon \ \ \|L(c \otimes z) - \phi(c)u\| < \epsilon \ \textit{ for all } \ c \in \mathcal{F} \quad (e\,8.262)$$

and L is T-$\mathcal{H}$-full.

Proof. We identify D with $C(\mathbb{T}, C)$. Let $f \in D_+ \setminus \{0\}$. There is positive number $b \ge 1$, $g \in D_+$ with $0 \le g \le b \cdot 1$ and $f_1 \in D_+ \setminus \{0\}$ with $0 \le f_1 \le 1$ such that

$$gfgf_1 = f_1. \quad (e\,8.263)$$

There is a point $t_0 \in \mathbb{T}$ such that $f_1(t_0) \ne 0$. There is $r > 0$ such that

$$\tau(f_1(t)) \ge \tau(f_1(t_0))/2$$

for all $\tau \in T(C)$ and for all t with $\mathrm{dist}(t, t_0) < r$.

Define $\Delta_0(f) = \inf\{\tau(f_1(t_0))/4 : \tau \in T(C)\} \cdot \Delta(r)$. There is an integer $n \geq 1$ such that

$$n \cdot \Delta_0(f) > 1. \tag{e 8.264}$$

Define $T(f) = (n, b)$. Put

$$\eta = \inf\{\Delta_0(f) : f \in \mathcal{H}\}/2 \text{ and } \epsilon_1 = \min\{\epsilon, \eta\}.$$

We claim that there exists an ϵ_1-$\mathcal{F} \cup \mathcal{H}$-multiplicative contractive completely positive linear map $L : D \to A$ such that

$$\|L(c \otimes 1) - \phi(c)\| < \epsilon \text{ for all } c \in \mathcal{F}, \ \|L(1 \otimes z) - u\| < \epsilon \text{ and} \tag{e 8.265}$$

$$|\tau \circ L(f_1) - \int_{\mathbb{T}} \tau(\phi(f_1(s)))d\mu_{\tau \circ \imath}(s)| < \eta \text{ for all } \tau \in \mathrm{T}(A) \tag{e 8.266}$$

and for all $f \in \mathcal{H}$. Otherwise, there exists a sequence of unitaries $\{u_n\} \subset U(A)$ for which $\mu_{\tau \circ \imath_n}(O_a) \geq \Delta(a)$ for all $\tau \in \mathrm{T}(A)$ and for any open balls O_a with radius $a \geq a_n$ with $a_n \to 0$, and for which

$$\lim_{n \to \infty} \|[\phi(c), u_n]\| = 0 \tag{e 8.267}$$

for all $c \in C$ and suppose for any sequence of contractive completely positive linear maps $L_n : D \to A$ with

$$\lim_{n \to \infty} \|L_n(ab) - L_n(a)L_n(b)\| = 0 \text{ for all } a, b \in D, \tag{e 8.268}$$

$$\lim_{n \to \infty} \|L_n(c \otimes f) - \phi(c)f(u_n)\| = 0, \tag{e 8.269}$$

for all $c \in C$, $f \in C(\mathbb{T})$ and

$$\liminf_n \{\max\{|\tau \circ L_n(f_1) - \int_{\mathbb{T}} \tau(\phi(f_1(s)))d\mu_{\tau \circ \imath_n}(s)| : f \in \mathcal{H}\}\} \geq \eta \tag{e 8.270}$$

for some $\tau \in \mathrm{T}(A)$, where $\imath_n : C(\mathbb{T}) \to D$ is defined by $\imath_n(f) = f(u_n)$ for $f \in C(\mathbb{T})$ (or no contractive completely positive linear maps L_n exists so that (e 8.268), (e 8.269) and (e 8.269)).

Put $A_n = A$, $n = 1, 2, ...,$ and $Q(A) = \prod_n A_n / \bigoplus_n A_n$. Let $\pi : \prod_n A_n \to Q(A)$ be the quotient map. Define a linear map $L' : D \to \prod_n A_n$ by $L(c \otimes 1) = \{\phi(c)\}$ and $L'(1 \otimes z) = \{u_n\}$. Then $\pi \circ L' : D \to Q(A)$ is a unital homomorphism. It follows from a theorem of Effros and Choi ([3]) that there exists a contractive completely positive linear map $L : D \to \prod_n A_n$ such that $\pi \circ L = \pi \circ L'$. Write $L = \{L_n\}$, where $L_n : D \to A_n$ is a contractive completely positive linear map. Note that

$$\lim_{n \to \infty} \|L_n(a)L_n(b) - L_n(ab)\| = 0 \text{ for all } a\, b \in D.$$

Fix $\tau \in \mathrm{T}(A)$, define $t_n : \prod_n A_n \to \mathbb{C}$ by $t_n(\{d_n\}) = \tau(d_n)$. Let t be a limit point of $\{t_n\}$. Then t gives a state on $\prod_n A_n$. Note that if $\{d_n\} \in \bigoplus_n A_n$, then $t_m(\{d_n\}) \to 0$. It follows that t gives a state $\bar{t}$ on $Q(A)$. Note that (by (e 8.269))

$$\bar{t}(\pi \circ L(c \otimes 1)) = \tau(\phi(c))$$

for all $c \in C$. It follows that

$$\bar{t}(\pi \circ L(f)) = \int_{\mathbb{T}} \bar{t}(\pi \circ L(f(s) \otimes 1))d\mu_{\bar{t} \circ \pi \circ L|_{1 \otimes C(\mathbb{T})}} = \int_{\mathbb{T}} \tau(\phi(f(s)))d\mu_{\bar{t} \circ \pi \circ L|_{1 \otimes C(\mathbb{T})}} \tag{e 8.271}$$

for all $f \in C(\mathbb{T}, C)$. Therefore, for a subsequence $\{n(k)\}$,

$$|\tau \circ L_{n(k)}(f_1) - \int_{\mathbb{T}} \tau(\phi(f_1(s)))d\mu_{\tau \circ \imath_{n(k)}}(s)| < \eta/2 \tag{e 8.272}$$

for all $f \in \mathcal{H}$. This contradicts with (e 8.270). Moreover, from this, it is easy to compute that

$$\mu_{\bar{t} \circ \pi \circ L|_{1 \otimes C(\mathbb{T})}}(O_a) \geq \Delta(a)$$

for all open balls O_a of t with radius $1 > a$. This proves the claim.

Note that

$$\int_{\mathbb{T}} \tau \circ \phi(f_1(s))d\mu_{\tau \circ \imath}(s) \geq (\tau(\phi(f_1(t_0)/2))) \cdot \Delta(r)$$

for all $\tau \in \mathrm{T}(A)$. It follows that

$$\tau(L(f_1)) \geq \inf\{t(f_1(t_0))/2 : t \in T(C)\} - \eta/2 \geq (4/3)\Delta_0(f) \tag{e 8.273}$$

for all $f \in \mathcal{H}$.

By Corollary 9.4 of [34], there exists a projection $e \in \overline{L(f_1)AL(f_1)}$ such that

$$\tau(e) \geq \Delta_0(f) \text{ for all } \tau \in \mathrm{T}(A). \tag{e 8.274}$$

It follows from (e 8.264) that there exists a partial isometry $w \in M_n(A)$ such that

$$w^* \mathrm{diag}(\overbrace{e, e, \cdots, e}^{n})w \geq 1_A.$$

Thus there $x_1, x_2, ..., x_n \in A$ with $\|x_i\| \leq 1$ such that

$$\sum_{i=1}^{n} x_i^* e x_i \geq 1. \tag{e 8.275}$$

Hence

$$\sum_{i=1}^{n} x_i^* g f g x_i \geq 1. \tag{e 8.276}$$

It then follows that there are $y_1, y_2, ..., y_n \in A$ with $\|y_i\| \leq b$ such that

$$\sum_{i=1}^{n} y_i^* f y_i = 1. \tag{e 8.277}$$

Therefore L is T-$\mathcal{H}$-full.

□

Lemma 8.2. *Let C be a unital separable amenable simple C^*-algebra with $TR(C) \leq 1$ satisfying the UCT. For $1/2 > \sigma > 0$, any finite subset $\mathcal{G}_0$ and any projections $p_1, p_2, ..., p_m \in C$. There is $\delta_0 > 0$, a finite subset $\mathcal{G} \subset C$ and a finite subset of projections $P_0 \subset C$ satisfying the following: Suppose that A is a unital simple C^*-algebra with $TR(A) \leq 1$, $\phi : C \to A$ is a unital homomorphism and $u \in U_0(A)$ is a unitary such that*

$$\|[\phi(c), u]\| < \delta < \delta_0 \text{ for all } c \in \mathcal{G} \cup \mathcal{G}_0 \text{ and } \mathrm{bott}_0(\phi, u)|_{\mathcal{P}_0} = \{0\}, \tag{e 8.278}$$

where $\mathcal{P}_0$ is the image of P_0 in $K_0(C)$. Then there exists a continuous path of unitaries $\{u(t) : t \in [0,1]\}$ in A with $u(0) = u$ and $u(1) = w$ such that

$$\begin{aligned} \|[\phi(c), u(t)]\| &< 3\delta \text{ for all } c \in \mathcal{G} \cup \mathcal{G}_0 \text{ and} && \text{(e 8.279)} \\ w_j \oplus (1 - \phi(p_j)) &\in CU(A), && \text{(e 8.280)} \end{aligned}$$

where $w_j \in U_0(\phi(p_j)A\phi(p_j))$ and

$$\|w_j - \phi(p_j)w\phi(p_j)\| < \sigma, \tag{e 8.281}$$

$j = 1, 2, ..., m$.

Moreover,

$$\mathrm{cel}(w_j \oplus (1 - \phi(p_j))) \le 8\pi + 1/4, \quad j = 1, 2, ..., m. \tag{e 8.282}$$

Proof. It follows from the combination of Theorem 4.8 and Theorem 4.9 of [9] and theorem 10.10 of [28] that one may write $C = \lim_{n\to\infty}(C_n, \psi_n)$, where each $C_n = \bigoplus_{j=1}^{R(n)} P_{n,j}C(X_{n,j}, M_{r(n,j)})P_{n,j}$ and where $P_{n,i} \in C(X_{n,i}, M_{r(n,i)})$ is a projection and $X_{n,i}$ is a connected finite CW complex of dimension no more than two with torsion free $K_1(C(X_{n,i}))$ and $K_0(C(X_{n,j})) = \mathbb{Z} \bigoplus \mathbb{Z}/s(j)\mathbb{Z}$ ($s(i) \ge 1$) and with positive cone $\{(0,0) \cup (m, x) : m \ge 1\}$ (when $s(j) = 1$, we mean $K_0(C(X_{n,j})) = \mathbb{Z}$), *or* $X_{n,i}$ is a connected finite CW complex of dimension three with $K_0(C(X_{n,i})) = \mathbb{Z}$ and torsion $K_1(C(X_{n,i}))$. Let $d(j)$ be the rank of $P_{n,j}$. It is known that one may assume that $d(j) \ge \prod_{j=1}^{R(n)} s(j) + 6$, $j = 1, 2, ..., R(n)$. This can be seen, for example, from Lemma 2.2, 2.3 (and the proof of Theorem 2.1) of [10].

Without loss of generality, we may assume that $\mathcal{G}_0 \subset \psi_{n,\infty}(C_n)$ and that there are projections $p_{i,0} \in C_n$ such that $\psi_{n,\infty}(p_{i,0}) = p_i$, $i = 1, 2, ..., m$. Choose, for each j, mutually orthogonal rank one projections $q_{j,0}^{(0)}, q_{j,1}^{(0)} \in P_{n,j}(C(X_{n,j}, M_{r(n,j)}))P_{n,j}$ such that

$$[q_{j,0}^{(0)}] = (1, 0) \text{ and } [q_{j,1}^{(0)}] = (1, \bar{1}) \in \mathbb{Z} \bigoplus \mathbb{Z}/s(j)\mathbb{Z},$$

or $q_{j,1}^{(0)} = 0$, if $K_0(C(X_{n,j})) = \mathbb{Z}$, $j = 1, 2, ..., R(n)$. Put $q'_{j,i} = \psi_{n,\infty}(q_{j,i}^{(0)})$ and $q_{j,i} = \phi(q'_{j,i})$, $i = 0, 1$ and $j = 1, 2, ..., R(n)$. Clearly, in C, p_k may be written as $W_j^* Q_j W_j$, where Q_j is a finite orthogonal sum of $q_{j,0}$ and $q_{j,1}$, $j = 1, 2, ..., R(n)\}$.

By choosing a sufficiently large $\mathcal{G}$ which contains $\mathcal{G}_0$ (and which contains Q_j, $q_{j,i}$ as well as W_j, among other elements) and sufficiently small $\delta_0 > 0$, one sees that it suffices to show the case that $\{p_1, p_2, ..., p_m\} \subset \{q_{j,0}, q_{j,1} : j = 1, 2, ..., R(n)\}$. Thus we obtain a finite subset $\mathcal{G}'$ and δ'_0 so that when $\mathcal{G} \supset \mathcal{G}'$ and $\delta_0 < \delta'_0$ one can make the assumption that $\{p_1, p_2, ..., p_m\} \subset \{q_{j,i}, i = 0, 1, \ \ j = 1, 2, ..., R(n)\}$. In particular, $\{q_{j,i}, i = 0, 1, \ \ j = 1, 2, ..., R(n)\} \subset \mathcal{G}'$.

Let $\mathcal{G}'_0 = \mathcal{G}_0 \cup \{q'_{j,0}, q'_{j,1} : j = 1, 2, ..., R(n)\}$. Fix $0 < \eta < \min\{\sigma/4, \delta'_0/2, 1/16\}$. Note that $P_{n,j}$ is locally trivial in $C(X_{n,j}, M_{r(n,j)})$. Since $TR(C) \le 1$, it has (SP) (see [19]). It is then easy to find a projection $e_j \in \psi_{n,\infty}(P_{n,j})C\psi_{n,\infty}(P_{n,j})$ and $B_j \cong M_{d(j)} \subset \psi_{n,\infty}(P_{n,j})C\psi_{n,\infty}(P_{n,j})$ with $1_{B_j} = e_j$ such that

$$\begin{aligned} \|[x, e_j]\| &< \eta \text{ for all } x \in \mathcal{G}'_0 && \text{(e 8.283)} \\ \mathrm{dist}(e_j x e_j, B_j) &< \eta \text{ for all } x \in \mathcal{G}'_0 \text{ and } e_j q'_{j,1} e_j, \, e_j q'_{j,0} e_j \ne 0, && \text{(e 8.284)} \end{aligned}$$

$j = 1, 2, ..., R(n)$. Furthermore, one may require that there is a projection $\bar{q}'_{j,i} \in B_j$ with rank one in B_j such that

$$\|\bar{q}'_{j,i} - e_j q'_{j,i} e_j\| < 2\eta, \quad i = 0, 1, \quad j = 1, 2, ..., R(n). \tag{e 8.285}$$

To simplify notation further, by replacing $q'_{j,i}$ by one of its nearby projections, we may assume that $q'_{j,i} = \bar{q}'_{j,i} + (q'_{j,i} - \bar{q}'_{j,i})$ and $q'_{j,i} \ge \bar{q}'_{j,i}$, $i = 0, 1$ and $j = 1, 2, ..., R(n)$. Since $s(j)[q'_{j,1}] = s(j)[q'_{j,0}]$, there is a unitary $Y_j \in P_{n,j}C(X_{n,j}, M_{r(n,j)})P_{n,j}$ such that

$$Y_j^* \mathrm{diag}(\overbrace{q'_{j,1}, q'_{j,1}, ..., q'_{j,1}}^{s(j)}, q'_{j,0}, q'_{j,0}, q'_{j,0})Y_j = \mathrm{diag}(\overbrace{q'_{j,0}, q'_{j,0}, ..., q'_{j,0}}^{s(j)+3}). \tag{e 8.286}$$

(Note that $d(j) \ge \prod_{j=1}^{R(n)} s(j) + 6$ and each $q'_{j,i}$ has rank one.)

Let $\{e_{i,k}^{(j)}\}$ be a matrix unit for B_j, $j = 1, 2, ..., R(n)$. We choose a finite subset $\mathcal{G}$ which contains $\mathcal{G}'_0$ as well as $\{e_{i,k}^{(j)}\}$, $\bar{q}'_{j,0}$, $\bar{q}'_{j,1}$ and $\{Y_j, Y_j^*\}$, $j = 1, 2, ..., R(n)$. Suppose that $v_{j,0} \in U_0(\phi(e_{1,1}^{(j)})A\phi(e_{1,1}^{(j)}))$ and

$$v_j = \mathrm{diag}(\overbrace{v_{j,0}, v_{j,0}, ..., v_{j,0}}^{d(j)}), \;\; j = 1, 2, ..., R(n). \tag{e 8.287}$$

Then

$$\phi(x)v_j = v_j\phi(x) \;\text{ for all }\; x \in B_j, \;\; j = 1, 2, ..., R(n). \tag{e 8.288}$$

Choose

$$\mathcal{P}_0 = \{[q'_{j,0}], [q'_{j,1}], [e_{i,i}^{(j)}], [\bar{q}'_{j,0}], [\bar{q}'_{j,1}], j = 1, 2, ..., R(n)\}.$$

Put $\bar{q}_{j,i} = \phi(\bar{q}'_{i,j})$, $i = 0, 1$ and $j = 1, 2, ..., R(n)$. We choose $\delta''_0 > 0$ such that $\mathrm{bott}_0(\phi, u)|_{\mathcal{P}_0}$ is well-defined which is zero and there is a unitary $u'_{j,i} \in U_0(\bar{q}_{j,i}A\bar{q}_{j,i})$ such that

$$\|u'_{j,i} - \bar{q}_{j,i}u\bar{q}_{j,i}\| < 2\delta''_0, \;\; i = 0, 1, \;\; j = 1, 2, ..., R(n),$$

whenever, $\|[\phi(c), u]\| < \delta''_0$ for all $c \in \mathcal{G}$.

Let $\delta_0 = \{1/32, \delta'_0/4, \delta''_0/4, \sigma/8\}$. Suppose that (e 8.278) holds for the above $\mathcal{G}$, $\mathcal{P}_0$ and $0 < \delta < \delta_0$. One obtains a unitary $u'_{j,i} \in U_0(\bar{q}_{j,i}A\bar{q}_{j,i})$ and a unitary $u''_{j,i} \in U_0((q_{j,i} - \bar{q}_{j,i})A(q_{j,i} - \bar{q}_{j,i}))$ such that

$$\|u_{j,i} - q_{j,i}uq_{j,i}\| < 2\delta, \tag{e 8.289}$$

where $u_{j,i} = u'_{j,i} + u''_{j,i}$, $i = 0, 1$ and $j = 1, 2, ..., R(n)$. It follows 3.4 (see also Theorem 6.6 of [28]) that there is $v_{j,0} \in U_0(\phi(e_{1,1}^{(j)})A\phi(e_{1,1}^{(j)}))$ such that

$$d(j)\overline{(v_{j,0} + (1 - \sum_{i=2}^{d(j)} \phi(e_{i,i}^{(j)})))} = \overline{u_{j,0}^*}, \;\text{ in }\; U_0(A)/CU(A), \;\; j = 1, 2, ..., R(n). \tag{e 8.290}$$

Put v_j as in (e 8.287). It follows from (e 8.290) that

$$\overline{(v_j \oplus (1 - \phi(e_j)))(u_{j,0} \oplus (1 - q_{j,0})} = \bar{1}, \tag{e 8.291}$$

$j = 1, 2, ..., R(n)$. Since $v_{j,0} \in U_0(\phi(e_j)A\phi(e_j))$, one has a continuous path of unitaries $\{v_{j,0}(t) : t \in [0, 1]\}$ such that $v_{j,0}(0) = \phi(e_{1,1}^{(j)})$ and $v_{j,0}(1) = v_{j,0}$, $j = 1, 2, ..., R(n)$. Put

$$v_j(t) = \mathrm{diag}(\overbrace{v_{j,0}(t), v_{j,0}(t), ..., v_{j,0}(t)}^{d(j)}), \;\; j = 1, 2, ..., R(n).$$

It follows that

$$\phi(x)v_j(t)=v_j(t)\phi(x) \ \text{ for all } \ x\in B_j \tag{e 8.292}$$

and $t\in[0,1]$, $j=1,2,...,R(n)$. Put

$$u(t)=(\sum_{j=1}^{R(n)}v_j(t)+(1-\sum_{j=1}^{R(n)}\phi(e_j))u \ \text{ for } \ t\in[0,1].$$

Note that, $u(0)=u$ and, if η is sufficiently small,

$$\|[\phi(c),\,u(t)]\|<2(\delta+\eta)<3\delta \ \text{ for all } \ c\in\mathcal{G}. \tag{e 8.293}$$

Put

$$w=u(1),\ \ w_{j,0}=(v_j\oplus(1-\phi(e_j))(u_{j,0}\oplus(1-q_{j,0})) \ \text{ and} \tag{e 8.294}$$
$$w_{j,1}=(v_j\oplus(1-\phi(e_j)))(u_{j,1}\oplus(1-q_{j,1}),w'_{j,i}=v_ju_{j,i}, \tag{e 8.295}$$

$i=0,1$, and $j=1,2,...,R(n)$. Define

$$\bar{w}_j=(v_j\oplus(1-\phi(e_j))(u_{j,0}\oplus u_{j,1}\oplus(1-q_{j,0}-q_{j,1})). \tag{e 8.296}$$

We have that

$$\bar{w}_jq_{j,i}=w_{j,i}q_{j,i}=v_ju_{j,i}=w'_{j,i}=q_{j,i}w_{j,i},$$

$i=0,1$ and $j=1,2,...,R(n)$. Note that, by (e 8.292),(e 8.283) and (e 8.285),

$$\|w_{j,i}q_{j,i}-q_{j,i}wq_{j,i}\|<\sigma \tag{e 8.297}$$

$i=0,1,\ \ j=1,2,...,R(n)$. By (e 8.291),

$$w_{j,0}\in CU(A),\ \ j=1,2,...,R(n). \tag{e 8.298}$$

It follows from Lemma 6.9 of [28] that

$$\mathrm{cel}(w_{j,0})\le 8\pi+1/4. \tag{e 8.299}$$

Put

$$E_j=1-\phi(Y_j^*\mathrm{diag}(\overbrace{q'_{j,1},q'_{j,1},...,q'_{j,1}}^{s(j)},q'_{j,0},q'_{j,0},q'_{j,0})Y_j).$$

It follows from (e 8.286) that in $U_0(A)/CU(A)$,

$$\begin{aligned}
\overline{w_{j,1}^{s(j)}w_{j,0}^3} &= \overline{\mathrm{diag}(\underbrace{\bar{w}_jq_{j,1},\bar{w}_jq_{j,1},...,\bar{w}_jq_{j,1}}_{s(j)},w_{j,0},w_{j,0},w_{j,0})\oplus E_j} && \text{(e 8.300)}\\
&= \overline{\phi(Y_j^*)\mathrm{diag}(\underbrace{\bar{w}_jq_{j,1},\bar{w}_jq_{j,1},...,\bar{w}_jq_{j,1}}_{s(j)},\bar{w}_jq_{j,0},\bar{w}_jq_{j,0},\bar{w}_jq_{j,0})\phi(Y_j)\oplus E_j} && \text{(e 8.301)}\\
&= \overline{\mathrm{diag}(\underbrace{w_{j,0},w_{j,0},...,w_{j,0}}_{s(j)+3})\oplus E_j}=\bar{1}, && \text{(e 8.302)}
\end{aligned}$$

where $j=1,2,...,R(n)$. By (e 8.298), the above implies that

$$\overline{w_{j,1}^{s(j)}}=\bar{1},\ \ j=1,2,...,R(n). \tag{e 8.303}$$

It follows from Theorem 6.11 that

$$w_{j,1} \in CU(A), \;\; j = 1, 2, ..., R(n). \tag{e 8.304}$$

It follows from Lemma 6.9 of [28] that

$$\mathrm{cel}(w_{j,1}) \le 8\pi + 1/4, \;\; j = 1, 2, ..., R(n).$$

□

Lemma 8.3. *Let C be a unital separable simple amenable C^*-algebra with $TR(C) \le 1$ satisfying the UCT. Let $\Delta : (0,1) \to (0,1)$ be a non-decreasing map. Then, for any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset C$, there exists $\delta > 0$, $\eta > 0$, a finite subset $\mathcal{G} \subset C$ and a finite subset $\mathcal{P} \subset \underline{K}(C)$ satisfying the following:*

For any unital simple C^*-algebra A with $TR(A) \le 1$, any unital homomorphism $\phi : C \to A$ and any unitary $u \in U(A)$ with

$$\|[\phi(f), u]\| < \delta, \;\; \mathrm{Bott}(\phi, u)|_{\mathcal{P}} = \{0\} \;\; \textit{and} \tag{e 8.305}$$

$$\mu_{\tau \circ \imath}(O_a) \ge \Delta(a) \;\; \textit{for all} \;\; a \ge \eta, \tag{e 8.306}$$

where $\imath : C(\mathbb{T}) \to A$ is defined by $\imath(f) = f(u)$ for all $f \in C(\mathbb{T})$, there exists a continuous path of unitaries $\{u(t) : t \in [0,1]\} \subset A$ such that

$$u(0) = u, \;\; u(1) = 1 \;\; \text{and} \;\; \|[\phi(f), u(t)]\| < \epsilon \tag{e 8.307}$$

for all $f \in \mathcal{F}$ and $t \in [0,1]$.

Proof. Let $\Delta_1 : (0,1) \to (0,1)$ be defined by $\Delta_1(a) = \Delta(a)/2$ for all $a \in (0,1)$. Put $D = C \otimes C(\mathbb{T})$. Let $T = N \times K : D_+ \setminus \{0\} \to \mathbb{N} \times \mathbb{R}_+ \setminus \{0\}$ be associated with Δ as in 8.1 and $T' = N' \times K' : D_+ \setminus \{0\} \to \mathbb{N} \times \mathbb{R}_+ \setminus \{0\}$ be associated with Δ_1 as in 8.1. Let $N_1 = \max\{N, N'\}$ and $K_1 = \max\{K, K'\}$. Define $T_0(h) = N_1(h) \times K_1(h)$ for $h \in D_+ \setminus \{0\}$.

Let $\epsilon > 0$ and $\mathcal{F} \subset C$ be a finite subset. Let $\mathcal{F}_1 = \{f \otimes g : f \in \mathcal{F} \cup \{1_C\}, \;\; g \in \{z, 1_{C(\mathbb{T})}\}\}$. Let $\delta_1 > 0$ (in place of δ), $\mathcal{G}_1 \subset D$ (in place of $\mathcal{G}$), $\mathcal{H}_0 \subset D_+ \setminus \{0\}$, $\mathcal{P}_1 \subset \underline{K}(D)$ (in place of $\mathcal{P}$) and $\mathcal{U} \subset U(D)$ be as required by 7.3 for $\epsilon/256$ (in place of ϵ), $\mathcal{F}_1$ and T_0 (in place of T). We may assume that $\delta_1 < \epsilon/256$.

To simplify notation, without loss of generality, we may assume that $\mathcal{H}_0$ is in the unit ball of D and $\mathcal{G}_1 = \{c \otimes g : c \in \mathcal{G}_1' \text{ and } g = 1_{C(\mathbb{T})}, g = z\}$, where $1_C \in \mathcal{G}_1'$ is a finite subset of C. Without loss of generality, we may assume that $\mathcal{U} = \mathcal{U}_1 \cup \{z_1, z_2, ..., z_n\}$, where $\mathcal{U}_1 \subset \{w \otimes 1_{C(\mathbb{T})} : w \in U(C)\}$ is a finite subset and $z_i = q_i \otimes z \oplus (1 - q_i) \otimes 1_{C(\mathbb{T})}$, $i = 1, 2, ..., n$ and $\{q_1, q_2, ..., q_n\} \subset C$ is a set of projections. We write $\underline{K}(D) = \underline{K}(C) \bigoplus \boldsymbol{\beta}(\underline{K}(C))$ (see 2.8). Without loss of generality, we may also assume that $\mathcal{P}_1 = \mathcal{P}_0 \cup \boldsymbol{\beta}(\mathcal{P}_2)$, where $\mathcal{P}_0, \mathcal{P}_2 \in \underline{K}(C)$ are finite subsets. Furthermore, we assume that $q_j \in \mathcal{G}_1'$ and $[q_j] \in \mathcal{P}_2$, $j = 1, 2, ..., n$. Let $\delta_0 > 0$ and let $\mathcal{G}_0 \subset C$ be finite subset such that there is a unital completely positive linear map $L' : D \to A$ such that

$$\|L'(c \otimes g) - \phi(c)g(u)\| < \delta_1/2 \;\; \text{for all} \;\; c \in \mathcal{G}_1' \;\; \text{and} \;\; g = 1 \;\; \text{or} \;\; g = z, \tag{e 8.308}$$

whenever there is a unitary $u \in A$ such that $\|[\phi(c), u]\| < \delta_0$ for all $c \in \mathcal{G}_0$. By applying 8.1, we may assume that, L' is T'-$\mathcal{H}_0$-full if, in addition,

$$\mu_{\tau \circ \imath}(O_a) \ge \Delta_1(a)$$

for all open balls O_a of $\mathbb{T}$ with radius $a \ge \eta_0$ for some $\eta_0 > 0$ and for all $\tau \in \mathrm{T}(A)$, where $\imath : C(\mathbb{T}) \to A$ is defined by $\imath(g) = g(u)$ for all $g \in C(\mathbb{T})$.

We may assume that

$$[L']|_{\mathcal{P}_0} = [L'']|_{\mathcal{P}_0} \tag{e 8.309}$$

for any pair of unital completely positive linear maps $L', L'' : C \otimes C(\mathbb{T}) \to A$ for which (e 8.308) holds for both L' and L'' and

$$L' \approx_{\delta_1} L'' \text{ on } \mathcal{G}_1'. \tag{e 8.310}$$

Choose an integer $K_0 \geq 1$ such that $[\frac{K_0-1}{\delta_1}] \geq 128/\delta_1$. In particular, $(8\pi+1)/[\frac{K_0-1}{\delta_1}] < \delta_1$.

Since $TR(C) \leq 1$, there is a projection $p \in C$ and a C^*-subalgebra $B = \bigoplus_{j=1}^k C(X_j, M_{r(j)})$, where $X_j = [0,1]$, or X_j is a point, with $1_B = p$ such that

$$\begin{aligned} \|[x,\, p]\| &< \min\{\epsilon/256, \delta_0/4, \delta_1/16\} \text{ for all } x \in \mathcal{G}_1' \cup \mathcal{G}_0 && \text{(e 8.311)}\\ \mathrm{dist}(pxp, B) &< \min\{\epsilon/256, \delta_0/4, \delta_1/16\} \text{ for all } x \in \mathcal{G}_1' \cup \mathcal{G}_0 \text{ and} && \text{(e 8.312)}\\ \tau(1-p) &< \min\{\delta_1/K_0, \Delta(\eta_0)/4, \delta_0/4\} \text{ for all } \tau \in T(C). && \text{(e 8.313)} \end{aligned}$$

We may also assume that there are projections $q_1', q_2', ..., q_n' \in (1-p)C(1-p)$ such that

$$\|q_i' - (1-p)q_i(1-p)\| < \min\{\epsilon/16, \delta_0/4, \delta_1/16\}, \ \ i = 1, 2, ..., n. \tag{e 8.314}$$

To simplify notation, without loss of generality, we may assume that p commutes with $\mathcal{G}' \cup \mathcal{G}_0$.

Moreover, we may assume that there is a unital completely positive linear map $L_{00} : C \to pCp \to B$ (first sending c to pcp then to B) such that

$$\|x - ((1-p)x(1-p) + L_{00}(x))\| < \min\{\epsilon/16, \delta_0/2, \delta_1/4\} \text{ for all } x \in \mathcal{G}_1. \tag{e 8.315}$$

Put $L_0'(c) = (1-p)c(1-p)$ and $L_0(c) = L_0'(c) + L_{00}(pcp)$ for all $c \in C$. We may further assume that $[L_{00}](\mathcal{P}_2)$ and $[L_0'](\mathcal{P}_2)$ are well-defined and

$$[L_0]|_{\mathcal{P}_0 \cup \mathcal{P}_2} = [\mathrm{id}_C]|_{\mathcal{P}_0 \cup \mathcal{P}_2}. \tag{e 8.316}$$

Put $\mathcal{P}_3 = [L_0'](\mathcal{P}_2) \cup \{[q_i'] : 1 \leq i \leq n\} \cup P_0$ and $\mathcal{P}_4 = [L_{00}](\mathcal{P}_2)$. From the above, $x = [L_0'](x) + [L_{00}](x)$ for $x \in \mathcal{P}_2$.

We also assume that

$$[L']|_{\boldsymbol{\beta}(\mathcal{P}_2 \cup \mathcal{P}_3 \cup \cap P_4)} = [L'']|_{\boldsymbol{\beta}(\mathcal{P}_2 \cup \mathcal{P}_3 \cup \cap P_4)} \tag{e 8.317}$$

for any pair of unital completely positive linear maps from $C \otimes C(\mathbb{T}) \to A$ such that

$$L_1 \approx_{\delta_2'} L_2 \text{ on } \mathcal{G}_2' \tag{e 8.318}$$

and items in (e 8.317) are well-defined for some $\delta_2' > 0$ and a finite subset $\mathcal{G}_2'$.

Let $\delta_2 > 0$ (in place of δ_0), $\mathcal{G}_2 \subset (1-p)C(1-p)$ and $P_0 \subset (1-p)C(1-p)$ be as required by 8.2 for $C = (1-p)C(1-p)$, $\sigma = \delta_1/16$, $\mathcal{G}_1' \cup \mathcal{G}_0$ (in place of $\mathcal{G}_0$) and $q_1', q_2', ..., q_n'$ (in place of $p_1, p_2, ..., p_m$). Note that we may assume that $P_0 \subset \mathcal{G}_2$.

Put $\mathcal{P}_3' = [L_0'](\mathcal{P}_2) \cup \{[q] : q \in P_0\}$. Note again that elements in $\mathcal{P}_3'$ are represented by elements in $(1-p)C(1-p)$. We may assume that

$$\mathrm{Bott}(\phi,\, u)|_{\mathcal{P}_3'} = \mathrm{Bott}(\phi, u')|_{\mathcal{P}_3'} \tag{e 8.319}$$

for any pair of unitaries u and u' in A for which

$$\|[\phi(c),\, u]\| < \min\{\delta_1, \delta_0\}, \ \ \|[\phi(c),\, u']\| < 2\min\{\delta_1, \delta_0\}$$

and for which there exists a continuous path of unitaries $\{W(t) : t \in [0,1]\} \subset (1-\phi(p))A(1-\phi(p))$ with

$$\|W(0) - (1-\phi(p))u(1-\phi(p))\| < \min\{\delta_1, \delta_0\} \text{ and } \quad (\text{e}\,8.320)$$
$$\|W(1) - (1-\phi(p))u'(1-\phi(p))\| < \min\{\delta_1, \delta_0\}, \quad (\text{e}\,8.321)$$

and

$$\|[\phi(c),\, W(t)]\| < \min\{\delta_1, \delta_0\}$$

for all $c \in \mathcal{G}_2$ and $t \in [0,1]$.

Write $p_i = 1_{C(X_i, M_{r(i)})} \in B$, $i = 1, 2, ..., k$. Let $\mathcal{F}_{0,i} = \{p_i x p_i : x \in \mathcal{F}\}$, $i = 1, 2, ..., k$. We may assume that $X_j = [0,1]$, $j = 1, 2, ..., k_0 \le k$ and X_j is a point for $i = k_0+1, k_0+2, ..., k$.

Put $D_j = C(X_j, M_{r(j)}) \otimes C(\mathbb{T})$. Define $T_i = N|_{(D_j)_+ \setminus \{0\}} \times 2R|_{(D_j)_+ \setminus \{0\}}$, $j = 1, 2, ..., k_0$. Let $\delta_{0,i} > 0$ (in place of δ), $\mathcal{H}_i \subset (D_i)_+ \setminus \{0\}$ and $\mathcal{G}_{0,i} \subset C(X_i, M_{r(i)})$ be required by 4.6 for $\epsilon/256k$ and $\mathcal{F}_{0,i}$ and T_i, $i = 1, 2, ..., k_0$. Let $\delta_{0,i} > 0$ (in place of δ), $\mathcal{G}_{0,i} \subset M_{r(i)}$ be required by 4.7 for $\epsilon/256k$ and $\mathcal{F}_{0,i}$, $i = k_0+1, k_0+2, ..., k$.

Denote by $\{e_{s,j}^{(i)}\}$ a matrix unit for $M_{r(i)}$, $i = 1, 2, ..., k$. Put

$$\bar{R} = \max\{N(h)R(h) : h \in \mathcal{H}_i, \ \ i = 1, 2, ..., k_0\}.$$

Let $\delta_3 = \min\{\epsilon/512, \delta_2/2, \delta_2'/2, \delta_1/16, \delta_{0,1}/2, \delta_{0,2}, ..., \delta_{0,k}/2\}$. Let $\mathcal{G}_3 = \mathcal{G}_2' \cup \mathcal{G}_1' \cup \mathcal{G}_2 \cup \{1-p, p\} \cup_{i=1}^{k_0} \mathcal{G}_{0,i}$. Let $\mathcal{H} = \mathcal{H}_0 \cup \{php : h \in \mathcal{H}_0\} \cup_{i=1}^{k_0} \mathcal{H}_i$ and let $\mathcal{P} = \mathcal{P}_2 \cup \mathcal{P}_3 \cup \mathcal{P}_4 \cup \{[1-p], [p], [e_{j,j}^{(i)}], [p_i], i = 1, 2, ..., k\}$.

It follows from 8.1 that there exists $\delta_4 > 0, \eta > 0$ and a finite subset $\mathcal{G}' \subset C$ satisfying the following: there exists a contractive completely positive linear map $L : D \to A$ which is T-$\mathcal{H}$-full such that

$$\|L(c \otimes 1) - \phi(c)\| < \delta_3/16k\bar{R} \text{ and } \|L(c \otimes z) - \phi(c)w\| < \delta_3/16k\bar{R} \text{ for all } c \in \mathcal{G}_3 \quad (\text{e}\,8.322)$$

and $[L]|_{\mathcal{P}_1 \cup \boldsymbol{\beta}(\mathcal{P}_2')}$ is well-defined, provided that $w \in A$ is a unitary with

$$\|[\phi(b),\, w]\| \ < \ 3\delta_4 \text{ for all } b \in \mathcal{G}' \text{ and} \quad (\text{e}\,8.323)$$
$$\mu_{\tau\circ\imath}(O_a) \ \ge \ \Delta(a) \quad (\text{e}\,8.324)$$

for all open balls O_a of $\mathbb{T}$ with radius $a \ge \eta$ for all $\tau \in \mathrm{T}(A)$, where $\imath : C(\mathbb{T}) \to A$ is defined by $\imath(f) = f(w)$ for $f \in C(\mathbb{T})$. We may assume that $\eta < \eta_0$ and $\delta_4 < \epsilon/256$.

Note that, for $h \in \mathcal{H}_i$,

$$L(h) \le L(\|h\|p_i) \le \|h\|L(p_i), \ \ i = 1, 2, ..., k. \quad (\text{e}\,8.325)$$

Therefore, we may assume that (with a smaller δ_4),

$$\|L(h) - \phi(p_i)L(h)\phi(p_i)\| < \delta_3/2k\bar{R} \quad (\text{e}\,8.326)$$

for any $h \in \mathcal{H}_i$, $i = 1, 2, ..., k_0$. We may also assume that

$$\|\phi(p_i)L(c \otimes z)\phi(p_i) - \phi(c)w_i'\| < \delta_3/16k\bar{R} \text{ for all } c \in p_i\mathcal{G}_3 p_i, \quad (\text{e}\,8.327)$$

provided that $w_i' \in U(\phi(p_i)A\phi(p_i))$ such that

$$\|w_i' - \phi(p_i)u\phi(p_i)\| < 3\delta_4, \ \ i = 1, 2, ..., k. \quad (\text{e}\,8.328)$$

For any function $g \in C(\mathbb{T})$ with $0 \le g \le 1$ and for any unitary $u \in U(A)$,

$$\tau(g(u)) = \tau(\phi(p)g(u)\phi(p)) + \tau((1-\phi(p))g(u)(1-\phi(p))) \text{ and } \tag{e 8.329}$$
$$\tau(\phi(p)g(u)\phi(p)) \ge \tau(g(u)) - \tau(1-\phi(p)) \text{ for all } \tau \in \mathrm{T}(A). \tag{e 8.330}$$

Thus, we may assume (by choosing smaller δ_4) that

$$\mu_{\tau\circ\imath'}(O_a) \ge \Delta(a)/2 \tag{e 8.331}$$

for all $a \ge \eta$ and $\tau \in \mathrm{T}(A)$, where $\imath' : C(\mathbb{T}) \to A$ is defined by $\imath'(f) = f(w)$ (for $f \in C(\mathbb{T})$) for any $w \in U(A)$ for which $w = w_0 \oplus w_1$, where $w_0 \in U((1-\phi(p))A(1-\phi(p)))$ and $w_1 \in U(\phi(p)A\phi(p))$, such that

$$\|w_1 - \phi(p)u\phi(p)\| < 2\delta_4, \tag{e 8.332}$$

where u and ϕ satisfy (e 8.322) and (e 8.323). Put $\delta = \min\{\delta_4/12, \delta_3/12\}$ and $\mathcal{G}_4 = \mathcal{G}' \cup \mathcal{G}_3$. Put $\mathcal{G} = \mathcal{G}_4 \cup \{(1-p)g(1-p) : g \in \mathcal{G}_3\} \cup \{e_{i,s}^{(0)}, [q_{j,0}]\}$.

Now suppose that ϕ and $u \in A$ satisfy the assumptions of the lemma for the above δ, η, $\mathcal{G}$ and $\mathcal{P}$. In particular, $u \in U_0(A)$. To simplify notation, without loss of generality, we may assume that all elements in $\mathcal{G}$ and in $\mathcal{H}$ have norm no more than 1.

By applying 8.2, one obtains a continuous path of unitaries $\{w_0(t) : t \in [0,1]\} \subset (1-\phi(p))A(1-\phi(p))$ and unitaries $w_j' \in U_0(\phi(q_j')A\phi(q_j'))$ such that

$$\|[\phi(c),\, w_0(t)]\| < 3\delta \text{ for all } c \in p\mathcal{G}p \text{ and} \tag{e 8.333}$$

for all $t \in [0,1]$,

$$\|w_0(0) - (1-\phi(p))u(1-\phi(p))\| < \delta_1/16, \tag{e 8.334}$$
$$\|w_j' - \phi(q_j')w_0(1)\phi(q_j')\| < \delta_1/16 \text{ and} \tag{e 8.335}$$
$$w_j' \oplus (1-\phi(p)-\phi(q_j')) \in CU((1-\phi(p))A(1-\phi(p))), \tag{e 8.336}$$

$j = 1, 2, ..., n$. Define $w = w_0(1) \oplus w_1$ for some unitary w_1 for which (e 8.332) holds.

We compute (by (e 8.305), (e 8.332) and (e 8.319)) that

$$\mathrm{Bott}(\phi,\, w)|_{\mathcal{P}} = \{0\}. \tag{e 8.337}$$

By (e 8.332), one also has that

$$\mu_{\tau\circ\imath'}(O_a) \ge \Delta_1(a) \text{ for all } \tau \in \mathrm{T}(A) \tag{e 8.338}$$

and for any open balls O_a of $\mathbb{T}$ with radius $a \ge \eta$, where $\imath' : C(\mathbb{T}) \to A$ is defined by $\imath'(g) = g(w)$ for all $g \in C(\mathbb{T})$.

Let $L : D \to A$ be a unital contractive completely positive linear map which satisfies (e 8.322). We may also assume that $[L]|_{\mathcal{P}}$ is well-defined

$$[L]|_{\mathcal{P}_0} = [\phi]|_{\mathcal{P}_0} \text{ and } [L]|_{\boldsymbol{\beta}(\mathcal{P})} = \{0\} \qquad \text{(by e 8.337)}. \tag{e 8.339}$$

There is a unital completely positive linear map $\Phi : (1-p)C(1-p) \otimes C(\mathbb{T}) \to (1-\phi(p))A(1-\phi(p))$ such that

$$\|\Phi(c \otimes g(z)) - \phi(c)g(w_0(1))\| < \delta_1/2 \tag{e 8.340}$$

for all $c \in \mathcal{G}_1' \cup \mathcal{G}_0$ and $g = 1_{C(\mathbb{T})}$ and $g = z$.

Define $L_1,\, L_2 : C\otimes C(\mathbb{T})\to A$ as follows:

$$L_1(c\otimes g(z)) \;=\; \Phi((1-p)c(1-p)\otimes g)\oplus \phi(p)L(c\otimes g)\phi(p)\;\;\text{and} \qquad (\mathrm{e}\,8.341)$$
$$L_2(c\otimes g) \;=\; \phi((1-p)c(1-p))g(1)\oplus \phi(p)L(c\otimes g)\phi(p) \qquad (\mathrm{e}\,8.342)$$

for all $c\in C$ and $g\in C(\mathbb{T})$. By (e 8.309), we compute that,

$$[L]|_{\mathcal{P}_0}=[L_1]|_{\mathcal{P}_0}=[L_2]|_{\mathcal{P}_0}. \qquad (\mathrm{e}\,8.343)$$

It is easy to see that that

$$[\phi(1-p)L_2\phi(1-p)]|_{\boldsymbol{\beta}(\mathcal{P}_2)}=\{0\}. \qquad (\mathrm{e}\,8.344)$$

One also has, by (e 8.310),

$$[L_2]|_{\boldsymbol{\beta}([L_{00}](\mathcal{P}_2))} \;=\; [L\circ L_{00}]|_{\boldsymbol{\beta}(\mathcal{P}_2))} \qquad (\mathrm{e}\,8.345)$$
$$\;=\; [L]|_{\boldsymbol{\beta}([L_{00}](\mathcal{P}_2))}=\mathrm{Bott}(\phi,u)|_{[L_{00}](\mathcal{P}_2)}=\{0\}. \qquad (\mathrm{e}\,8.346)$$

Combining (e 8.344) and (e 8.346), one obtains that

$$[L_2]|_{\boldsymbol{\beta}(\mathcal{P}_2)}=\{0\}. \qquad (\mathrm{e}\,8.347)$$

From (e 8.319), one computes that

$$[L_1]|_{\boldsymbol{\beta}(\mathcal{P}_2)}=[L]|_{\boldsymbol{\beta}(\mathcal{P}_2)}=\mathrm{Bott}(\phi,\,u)|_{\mathcal{P}_2}=\{0\}. \qquad (\mathrm{e}\,8.348)$$

It follows that

$$[L_1]|_{\mathcal{P}_1}=[L]|_{\mathcal{P}_1}. \qquad (\mathrm{e}\,8.349)$$

It is routine to check that

$$|\tau\circ L(g)-\tau\circ L_1(g)|<\delta_1\;\;\text{for all}\;\; g\in\mathcal{G}_1\;\;\text{for all}\;\;\tau\in \mathrm{T}(A). \qquad (\mathrm{e}\,8.350)$$

If $v\in\mathcal{U}_1$, since $\|\phi(v)-L_1(v\otimes 1)\|<\delta_1/2$ and $\|\phi(v)-L_2(v\otimes 1)\|<\delta_1/2$, it follows that

$$\mathrm{dist}(L_1^{\ddagger}(\bar{v}),L_2^{\ddagger}(\bar{v}))<\delta_1. \qquad (\mathrm{e}\,8.351)$$

If $\zeta_j=q_j\otimes z,\ j=1,2,...,n$, by (e 8.334), (e 8.335) and (e 8.336), by the choice of K_0 and by applying 3.1, one has that

$$\mathrm{dist}((L_1^{\ddagger}(\bar{\zeta}_j),L_2^{\ddagger}(\bar{\zeta}_j))<\delta_1. \qquad (\mathrm{e}\,8.352)$$

Note also that, by (e 8.338) and by 8.1, both L_1 and L_2 are T_0-$\mathcal{H}_0$-full. It then follows from (e 8.346), (e 8.347), (e 8.351), (e 8.352) and 7.3 that there exists a unitary $W\in U(A)$ such that

$$\mathrm{ad}\,W\circ L_2\approx_{\epsilon/256} L_1\;\;\text{on}\;\;\mathcal{F}_1. \qquad (\mathrm{e}\,8.353)$$

We may assume that

$$\|u_i-\phi(p_i)u\phi(p_i)\|<2\delta\;\;\text{and}\;\; w_1=\sum_{i=1}^{k}u_i \qquad (\mathrm{e}\,8.354)$$

for some $u_i\in U(\phi(p_i)A\phi(p_i)),\ i=1,2,...,R(n)$ and

$$u_i\in U_0(\phi(p_i)A\phi(p_i)),\;\; i=1,2,...,k \qquad (\mathrm{e}\,8.355)$$

(since $\mathrm{Bott}(\phi,\, u)|_{\mathcal{P}} = \{0\}$). There is a positive element $a_i \in \phi(p_i)A\phi(p_i)$ such that

$$a_i L(p_i) a_i = \phi(p_i) \ \text{ and } \ \|a_i - \phi(p_i)\| < \delta_3/8k\bar{R}, \ \ i = 1, 2, ..., k. \tag{e 8.356}$$

Let $\Psi_i : D_i \to \phi(p_i)A\phi(p_i)$ be defined by $\Psi_i(a) = a_i\phi(p_i)L(a)\phi(p_i)a_i$ for all $a \in D_i$, $i = 1, 2, ..., k$. Thus

$$\|\Psi_i(h) - \phi(p_i)L(h)\phi(p_i)\| < \delta_3/4k\bar{R} \tag{e 8.357}$$

for all $h \in \mathcal{H}_i$, $i = 1, 2, ..., k$. Note also that (by e 8.327))

$$\|\Psi_i(c \otimes 1) - \phi(c)\| < \delta + \delta_3/4k\bar{R} \ \text{ and } \ \|L_i(c \otimes z) - \phi(c)u_i\| < \delta_3/4k\bar{R} \tag{e 8.358}$$

for all $c \in \mathcal{G}_{0,i}$, $i = 1, 2, ..., k$. Note also that

$$\mathrm{bott}_0(\phi|_{C(X_i, M_{r(i)})},\, u_i) = \{0\}, \ \ i = 1, 2, ..., k. \tag{e 8.359}$$

Furthermore, for each $h \in \mathcal{H}_i$, there exist $x_1(h), x_2(h), ..., x_{N(h)}(h)$ with and $\|x_j\| \le R(h)$, $j = 1, 2, ..., N(h)$ such that

$$\sum_{j=1}^{N(h)} x_j(h)^* L(h) x_j(h) = 1_A. \tag{e 8.360}$$

It follows from (e 8.357) that

$$\|\sum_{j=1}^{N(h)} x_j(h)^*\Psi_i(h)x_j(h) - 1_A\| < N(h)R(h)(\frac{\delta_3}{4k\bar{R}}) < \delta_3/4k \tag{e 8.361}$$

Therefore that there exists $y(h) \in A_+$ with $\|y(h)\| \le \sqrt{2}$ such that

$$\sum_{j=1}^{N(h)} y(h)(x_j(h))^*\Phi_i(h)(x_j(h))y(h) = \phi(p_i). \tag{e 8.362}$$

It follows that Φ_i is T_i-$\mathcal{H}_i$-full, $i = 1, 2, ..., k$.

It follows from 4.6 and 4.7 that there is a continuous path of unitaries $\{u_i(t) : t \in [0, 1]\} \subset \phi(p_i)A\phi(p_i)$ such that

$$u_i(0) = u_i, \ \ u_i(1) = p_i \ \text{ and } \tag{e 8.363}$$

$$\|[\Psi_i(c),\, u_i(t)]\| < \epsilon/k256 \ \text{ for all } \ c \in \mathcal{F}_{0,i} \tag{e 8.364}$$

and for all $t \in [0, 1]$, $i = 1, 2, ..., k$.

Define a continuous path of unitaries $\{z(t) : t \in [0, 1]\} \subset A$ by

$$z(t) = (1 - \phi(p)) \oplus \sum_{i=1}^{k} u_i(t) \quad \text{for all } \ t \in [0, 1].$$

Then $z(0) = (1 - \phi(p)) + \sum_{i=1}^{k} u_i$ and $z(1) = 1_A$. By (e 8.364), (e 8.357) and (e 8.322),

$$\|[\phi(c),\, z(t)]\| < \epsilon/128 \ \text{ for all } \ c \in \mathcal{F}. \tag{e 8.365}$$

Define $u'(t) = (w_0(t)w_0(1)^* \oplus (1-\phi(p)))W^*z(t)W$ for $t \in [0,1]$. Then $u'(1) = 1_A$ and we estimate by (e 8.332), (e 8.322), (e 8.353), (e 8.340) and (e 8.332) again that

$$\begin{aligned} u'(0) &\approx_{2\delta_4+\delta_3/2\bar{R}} (w_0(0)w_0(1)^* \oplus (1-\phi(p)))W^*L_2(1\otimes z)W && \text{(e 8.366)}\\ &\approx_{\epsilon/256} (w_0(0)w_0(1)^* \oplus (1-\phi(p))L_1(1\otimes z) && \text{(e 8.367)}\\ &\approx_{\delta_1/2+\delta_3/2\bar{R}} (w_0(0) \oplus (1-\phi(p))((1-\phi(p)) \oplus w_1) && \text{(e 8.368)}\\ &\approx_{\delta_1/16+2\delta_4} (1-\phi(p))u(1-\phi(u)) \oplus \phi(p)u\phi(u). && \text{(e 8.369)} \end{aligned}$$

It follows that

$$\|u'(0) - u\| < \epsilon/8. \tag{e 8.370}$$

Moreover, by (e 8.353), $W^*\phi(c)W \approx_{\epsilon/256} \phi(c)$ for all $c \in \mathcal{F}$. It follows that

$$\|[\phi(c),\, u'(t)]\| < \epsilon/2 \text{ for all } c \in \mathcal{F} \text{ and } t \in [0,1]. \tag{e 8.371}$$

The lemma follows when one connects u with $u'(0)$ with a continuous path of length no more than $(\epsilon/8)\pi$.

□

Theorem 8.4. *Let C be a unital separable amenable simple C^*-algebra with $TR(C) \le 1$ which satisfies the UCT. For any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset C$, there exists $\delta > 0$, a finite subset $\mathcal{G} \subset C$ and a finite subset $\mathcal{P} \subset \underline{K}(C)$ satisfying the following:*

Suppose that A is a unital simple C^*-algebra with $TR(A) \le 1$, suppose that $\phi : C \to A$ is a unital homomorphism and $u \in U(A)$ such that

$$\|[\phi(c),\, u]\| < \delta \textit{ for all } c \in \mathcal{G} \textit{ and } \mathrm{Bott}(\phi, u)|_{\mathcal{P}} = 0. \tag{e 8.372}$$

Then there exists a continuous and piece-wise smooth path of unitaries $\{u(t) : t \in [0,1]\}$ such that

$$u(0) = u,\;\; u(1) = 1 \text{ and } \|[\phi(c),\, u(t)]\| < \epsilon \textit{ for all } c \in \mathcal{F} \tag{e 8.373}$$

and for all $t \in [0,1]$.

Proof. Fix $\epsilon > 0$ and a finite subset $\mathcal{F} \subset C$. Let $\delta_1 > 0$ (in place of δ), $\eta > 0$, $\mathcal{G}_1 \subset C$ (in place of $\mathcal{G}$) be a finite subset and $\mathcal{P} \subset \underline{K}(C)$ be finite subset as required by 8.3 for ϵ, $\mathcal{F}$ and $\Delta = \Delta_{00}$. We may assume that $\delta_1 < \epsilon$.

Let $\delta = \delta_1/2$. Suppose that ϕ and u satisfy the conditions in the theorem for the above δ, $\mathcal{G}$ and $\mathcal{P}$. It follows from 5.9 that there is a continuous path of unitaries $\{v(t) : t \in [0,1]\} \subset U(A)$ such that

$$v(0) = u,\;\; v(1) = u_1 \text{ and } \|[\phi(c), v(t)]\| < \delta_1 \tag{e 8.374}$$

for all $c \in \mathcal{G}_1$ and for all $t \in [0,1]$, and

$$\mu_{\tau\circ\imath}(O_a) \ge \Delta(a) \text{ for all } \tau \in \mathrm{T}(A) \tag{e 8.375}$$

and for all open balls of radius $a \ge \eta$.

By applying 8.3, there is a continuous path of unitaries $\{w(t) : t \in [0,1]\} \subset A$ such that

$$w(0) = u_1,\;\; w(1) = 1 \text{ and } \|[\phi(c),\, w(t)]\| < \epsilon \tag{e 8.376}$$

for all $c \in \mathcal{F}$ and $t \in [0,1]$. Put

$$u(t) = v(2t) \text{ for all } t \in [0,1/2) \text{ and } u(t) = w(2t-1/2) \text{ for all } t \in [1/2,1].$$

Remark 6.4 shows that we can actually require, in addition, the path is piece-wise smooth.

□

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